

# ECON 337901

# FINANCIAL ECONOMICS

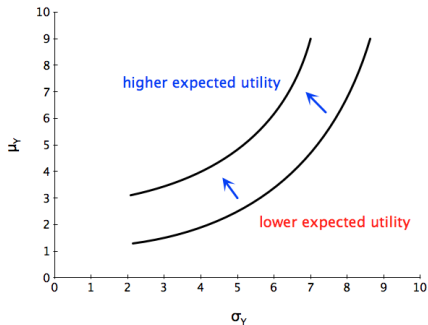
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# Justifying Mean-Variance Utility



Upward-sloping indifference curves are implied by mean-variance utility. But what does the constraint look like in this diagram? To see, we need to consider the gains from diversification.

## The Gains From Diversification

One of the most important lessons that we can take from modern portfolio theory involves the gains from diversification.

To see where these gains come from, consider forming a portfolio from two risky assets:

$\tilde{r}_1, \tilde{r}_2 =$  random returns

$\mu_1, \mu_2 =$  expected returns

$\sigma_1, \sigma_2 =$  standard deviations

Assume  $\mu_1 > \mu_2$  and  $\sigma_1 > \sigma_2$  to create a trade-off between expected return and risk **if** the investor must choose between one or the other.

## The Gains From Diversification

If  $w$  is the fraction of initial wealth allocated to asset 1 and  $1 - w$  is the fraction of initial wealth allocated to asset 2, the random return  $\tilde{r}_P$  on the portfolio is

$$\tilde{r}_P = w\tilde{r}_1 + (1 - w)\tilde{r}_2$$

and the expected return  $\mu_P$  on the portfolio is

$$\begin{aligned}\mu_P &= E[w\tilde{r}_1 + (1 - w)\tilde{r}_2] \\ &= wE(\tilde{r}_1) + (1 - w)E(\tilde{r}_2) \\ &= w\mu_1 + (1 - w)\mu_2\end{aligned}$$

## The Gains From Diversification

$$\mu_P = w\mu_1 + (1 - w)\mu_2$$

The expected return on the portfolio is a weighted average of the expected returns on the individual assets.

Since  $\mu_1 > \mu_2$ ,  $\mu_P$  can range from  $\mu_2$  up to  $\mu_1$  as  $w$  increases from zero to one. Even higher (or lower) expected returns are possible if short selling is allowed.

## The Gains From Diversification

But now let's calculate the variance of the random portfolio return

$$\tilde{r}_P = w\tilde{r}_1 + (1 - w)\tilde{r}_2$$

$$\begin{aligned}\sigma_P^2 &= E[(\tilde{r}_P - \mu_P)^2] \\ &= E\{[w\tilde{r}_1 + (1 - w)\tilde{r}_2 - w\mu_1 - (1 - w)\mu_2]^2\} \\ &= E\{[w(\tilde{r}_1 - \mu_1) + (1 - w)(\tilde{r}_2 - \mu_2)]^2\} \\ &= E[w^2(\tilde{r}_1 - \mu_1)^2 + (1 - w)^2(\tilde{r}_2 - \mu_2)^2 \\ &\quad + 2w(1 - w)(\tilde{r}_1 - \mu_1)(\tilde{r}_2 - \mu_2)]\end{aligned}$$

## The Gains From Diversification

$$\sigma_P^2 = E[w^2(\tilde{r}_1 - \mu_1)^2 + (1 - w)^2(\tilde{r}_2 - \mu_2)^2 + 2w(1 - w)(\tilde{r}_1 - \mu_1)(\tilde{r}_2 - \mu_2)]$$

$$\sigma_P^2 = w^2 E[(\tilde{r}_1 - \mu_1)^2] + (1 - w)^2 E[(\tilde{r}_2 - \mu_2)^2] + 2w(1 - w) E[(\tilde{r}_1 - \mu_1)(\tilde{r}_2 - \mu_2)]$$

## The Gains From Diversification

In probability theory, the **covariance** between two random variables  $X_1$  and  $X_2$  is defined as

$$\sigma(X_1, X_2) = E\{[X_1 - E(X_1)][X_2 - E(X_2)]\}$$

and the **correlation** between  $X_1$  and  $X_2$  is defined as

$$\rho(X_1, X_2) = \frac{\sigma(X_1, X_2)}{\sigma(X_1)\sigma(X_2)}$$



## The Gains From Diversification

The covariance

$$\sigma(X_1, X_2) = E\{[X_1 - E(X_1)][X_2 - E(X_2)]\}$$

is positive if

$$X_1 - E(X_1) \text{ and } X_2 - E(X_2)$$

tend to have the same sign, negative

$$X_1 - E(X_1) \text{ and } X_2 - E(X_2)$$

tend to have opposite signs, and zero if

$$X_1 - E(X_1) \text{ and } X_2 - E(X_2)$$

show no tendency to have the same or opposite signs.

## The Gains From Diversification

Mathematically, therefore, the covariance

$$\sigma(X_1, X_2) = E\{[X_1 - E(X_1)][X_2 - E(X_2)]\}$$

measures the extent to which the two random variables tend to move together.

Economically, buying two assets with returns that are imperfectly, and especially, negatively correlated is like buying insurance: one return will be high when the other is low and vice versa, reducing the overall risk of the portfolio.

# The Gains From Diversification

The correlation

$$\rho(X_1, X_2) = \frac{\sigma(X_1, X_2)}{\sigma(X_1)\sigma(X_2)}$$

has the same sign as the covariance, and is therefore also a measure of co-movement.

But “scaling” the covariance by the two standard deviations makes the correlation range between  $-1$  and  $1$ :

$$-1 \leq \rho(X_1, X_2) \leq 1$$

## The Gains From Diversification

Hence

$$\begin{aligned}\sigma_P^2 &= w^2 E[(\tilde{r}_1 - \mu_1)^2] + (1 - w)^2 E[(\tilde{r}_2 - \mu_2)^2] \\ &\quad + 2w(1 - w)E[(\tilde{r}_1 - \mu_1)(\tilde{r}_2 - \mu_2)]\end{aligned}$$

implies

$$\begin{aligned}\sigma_P^2 &= w^2 \sigma_1^2 + (1 - w)^2 \sigma_2^2 + 2w(1 - w)\sigma_{12} \\ &= w^2 \sigma_1^2 + (1 - w)^2 \sigma_2^2 + 2w(1 - w)\sigma_1 \sigma_2 \rho_{12}\end{aligned}$$

where

$\sigma_{12}$  = the covariance between  $\tilde{r}_1$  and  $\tilde{r}_2$

$\rho_{12}$  = the correlation between  $\tilde{r}_1$  and  $\tilde{r}_2$

## The Gains From Diversification

This is the source of the gains from diversification: the expected portfolio return

$$\mu_P = w\mu_1 + (1 - w)\mu_2$$

is a weighted average of the expected returns on the individual asset returns, but the standard deviation of the portfolio return

$$\sigma_P = [w^2\sigma_1^2 + (1 - w)^2\sigma_2^2 + 2w(1 - w)\sigma_1\sigma_2\rho_{12}]^{1/2}$$

is **not** a weighted average of the standard deviations of the returns on the individual assets and can be reduced by choosing a mix of assets ( $0 < w < 1$ ) when  $\rho_{12}$  is less than one and, especially, when  $\rho_{12}$  is negative.

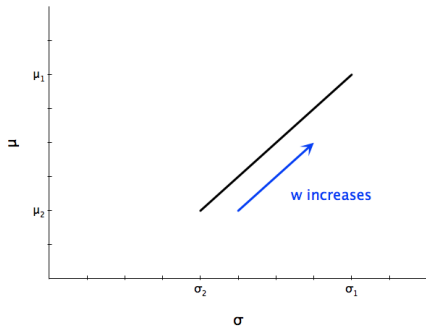
## The Gains From Diversification

To see more specifically how this works, start with the case where  $\rho_{12} = 1$  so that the individual asset returns are perfectly correlated. This is the one case in which there are no gains from diversification. With  $\rho_{12} = 1$ ,

$$\begin{aligned}\sigma_P &= [w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_1\sigma_2\rho_{12}]^{1/2} \\ &= [w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_1\sigma_2]^{1/2} \\ &= \{[w\sigma_1 + (1-w)\sigma_2]^2\}^{1/2} \\ &= |w\sigma_1 + (1-w)\sigma_2|.\end{aligned}$$

In this special case, the standard deviation of the return on the portfolio **is** a weighted average of the standard deviations of the returns on the individual assets.

## The Gains From Diversification



When  $\rho_{12} = 1$ , so that individual asset returns are perfectly correlated, there are no gains from diversification.

## The Gains From Diversification

Next, let's consider the opposite extreme, in which  $\rho_{12} = -1$  so that the individual asset returns are perfectly, but negatively, correlated:

$$\begin{aligned}\sigma_P &= [w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_1\sigma_2\rho_{12}]^{1/2} \\ &= [w^2\sigma_1^2 + (1-w)^2\sigma_2^2 - 2w(1-w)\sigma_1\sigma_2]^{1/2} \\ &= \{[w\sigma_1 - (1-w)\sigma_2]^2\}^{1/2} \\ &= |w\sigma_1 - (1-w)\sigma_2|.\end{aligned}$$

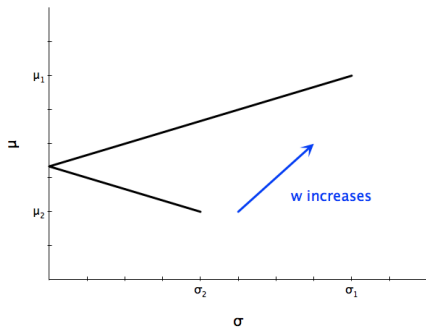
In this special case, the setting

$$w = \frac{\sigma_2}{\sigma_1 + \sigma_2}$$

creates a “synthetic” risk free portfolio!



## The Gains From Diversification



When  $\rho_{12} = -1$ , so that individual asset returns are perfectly, but negatively correlated, risk can be eliminated via diversification.

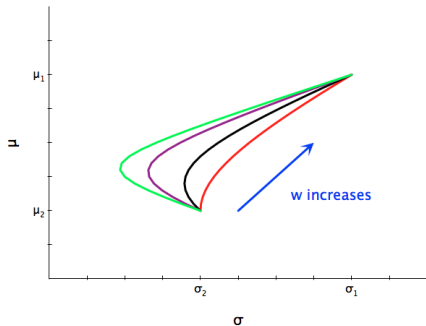
## The Gains From Diversification

$$\mu_P = w\mu_1 + (1 - w)\mu_2$$

$$\sigma_P = [w^2\sigma_1^2 + (1 - w)^2\sigma_2^2 + 2w(1 - w)\sigma_1\sigma_2\rho_{12}]^{1/2}$$

In all intermediate cases, there will still be gains from diversification. These gains will become stronger as  $\rho_{12}$  declines from 1 to  $-1$ .

## The Gains From Diversification



As  $\rho_{12}$  decreases from 0.5 to 0 to -0.5 to -0.75, the gains from diversification strengthen.

## The Gains from Diversification

$$\tilde{r}_p = w\tilde{r}_1 + (1 - w)\tilde{r}_2$$

$$\mu_P = w\mu_1 + (1 - w)\mu_2$$

$$\sigma_P = [w^2\sigma_1^2 + (1 - w)^2\sigma_2^2 + 2w(1 - w)\sigma_1\sigma_2\rho_{12}]^{1/2}$$

PS11: Suppose  $\mu_1 = 8$ ,  $\mu_2 = 4$ ,  $\sigma_1 = 8$ , and  $\sigma_2 = 4$ .

Calculate  $\mu_P$  and  $\sigma_P$  for various values of  $w$  when  $\rho_{12} = 0$  and  $\rho_{12} = -0.5$ . The gains from diversification are strongest when  $\rho_{12}$  is negative, but still present whenever  $\rho_{12} < 1$ .