

ECON 337901

FINANCIAL ECONOMICS

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Risk Aversion and Portfolio Allocation

Danthine and Donaldson (3rd ed., p.41) report that in the United States, 1889-2010, average real (inflation-adjusted) returns on stocks and risk-free bonds are

$$E(\tilde{r}) = 0.075 \text{ (7.5 percent per year)}$$

$$r_f = 0.011 \text{ (1.1 percent per year)}$$

The **equity risk premium** of $E(\tilde{r}) - r_f = 0.064$ (6.4 percent) is not only positive, it is huge.

Risk Aversion and Portfolio Allocation

Using updated data (1871-2023) on the S&P 500, adjusted for dividends and inflation from Robert Shiller's webpage:

<http://www.econ.yale.edu/~shiller/data.htm>

$$E(\tilde{r}) = 0.085$$

$$\sigma(\tilde{r}) = 0.178$$

$$E(\tilde{r}) - 2\sigma(\tilde{r}) = -0.271$$

$$E(\tilde{r}) + 2\sigma(\tilde{r}) = 0.441$$

$$r_{1931} = -0.380$$

$$r_{1933} = 0.531$$

$$r_{2008} = -0.356$$

$$r_{2009} = 0.298$$

$$r_{2020} = 0.162$$

$$r_{2021} = 0.137$$

$$r_{2022} = -0.173$$

$$r_{2023} = 0.197$$

Risk Aversion and Portfolio Allocation

Let's now put our framework of decision-making under uncertainty to use.

Consider a risk-averse investor with vN-M expected utility who divides his or her initial wealth Y_0 into an amount a allocated to a risky asset – say, the stock market – and an amount $Y_0 - a$ allocated to a safe asset – say, a bank account or a government bond.

Risk Aversion and Portfolio Allocation

Y_0 = initial wealth

a = amount allocated to stocks

\tilde{r} = random return on stocks

r_f = risk-free return

\tilde{Y}_1 = terminal wealth

$$\begin{aligned}\tilde{Y}_1 &= (1 + r_f)(Y_0 - a) + a(1 + \tilde{r}) \\ &= Y_0(1 + r_f) + a(\tilde{r} - r_f)\end{aligned}$$

Risk Aversion and Portfolio Allocation

The investor chooses a to maximize expected utility:

$$\max_a E[u(\tilde{Y}_1)] = \max_a E\{u[Y_0(1 + r_f) + a(\tilde{r} - r_f)]\}$$

The first-order condition is

$$E\{u'[Y_0(1 + r_f) + a^*(\tilde{r} - r_f)](\tilde{r} - r_f)\} = 0.$$

Note: we are allowing the investor to sell stocks short ($a^* < 0$) or to buy stocks on margin ($a^* > Y_0$) if he or she desires.

Risk Aversion and Portfolio Allocation

The following results were proven by Kenneth Arrow in “The Theory of Risk Aversion,” published in the 1971 volume *Essays in the Theory of Risk-Bearing* and reprinted in 1983 in volume 3 of the *Collected Papers of Kenneth J. Arrow* (Harvard University Press).

Risk Aversion and Portfolio Allocation

Theorem If the Bernoulli utility function u is increasing and concave, then

$$a^* > 0 \text{ if and only if } E(\tilde{r}) > r_f$$

$$a^* = 0 \text{ if and only if } E(\tilde{r}) = r_f$$

$$a^* < 0 \text{ if and only if } E(\tilde{r}) < r_f$$

Thus, a risk-averse investor will **always** allocate at least some funds to the stock market if the expected return on stocks exceeds the risk-free rate.

Risk Aversion and Portfolio Allocation

Arrow also showed that a^* rises when either $R_A(Y_0)$ or $R_R(Y_0)$ goes down.

If there are investors with the same initial wealth but different coefficients of risk aversion, the investor who is less risk averse will hold more stocks.

Risk Aversion and Portfolio Allocation

Arrow also showed that if an investor has:

1. Decreasing absolute risk aversion, then a^* rises with Y_0 .
2. Constant absolute risk aversion, then a^* does not depend on Y_0 .
3. Increasing absolute risk aversion, then a^* falls with Y_0 .

Intuitively, if $R_A(Y_0)$ is constant (does not depend on Y_0), then the investor finds his or her “optimal absolute bet” a^* and sticks with it, even as income goes up.

Risk Aversion and Portfolio Allocation

Arrow also showed that if an investor has:

1. Decreasing relative risk aversion, then a^*/Y_0 rises with Y_0 .
2. Constant relative risk aversion, then a^*/Y_0 does not depend on Y_0 .
3. Increasing relative risk aversion, then a^*/Y_0 falls with Y_0 .

Intuitively, if $R_R(Y_0)$ is constant (does not depend on Y_0), then the investor finds his or her “optimal bet” a^*/Y_0 as a fraction of income, and lets a^* rise proportionally as his or her income goes up.

Risk Aversion and Portfolio Allocation

As an example, suppose $u(Y) = \ln(Y)$, as suggested by Daniel Bernoulli. Recall that for this utility function, $u'(Y) = 1/Y$. Then assume that stock returns can either be good or bad:

$$\tilde{r} = \begin{cases} r_G & \text{with probability } \pi \\ r_B & \text{with probability } 1 - \pi \end{cases}$$

where $r_G > r_f > r_B$ defines the “good” and “bad” states and

$$\pi r_G + (1 - \pi)r_B > r_f,$$

so that $E(\tilde{r}) > r_f$ and the investor will choose $a^* > 0$.

Risk Aversion and Portfolio Allocation

The problem

$$\max_a E\{u[Y_0(1 + r_f) + a(\tilde{r} - r_f)]\}$$

specializes to

$$\begin{aligned} \max_a \quad & \pi \ln[Y_0(1 + r_f) + a(r_G - r_f)] \\ & + (1 - \pi) \ln[Y_0(1 + r_f) + a(r_B - r_f)] \end{aligned}$$

Risk Aversion and Portfolio Allocation

Problem Set 10, Question 1: Specialize the problem

$$\begin{aligned} \max_a \quad & \pi \ln[Y_0(1 + r_f) + a(r_G - r_f)] \\ & + (1 - \pi) \ln[Y_0(1 + r_f) + a(r_B - r_f)] \end{aligned}$$

by setting $Y_0 = 100$, $r_f = 0.10$, $r_G = 0.30$, $r_B = 0.05$, and $\pi = 1 - \pi = 1/2$.

The problem becomes

$$\max_a (1/2) \ln(110 + 0.20a) + (1/2) \ln(110 - 0.05a)$$

To find a^* , differentiate with respect to a using the chain rule and set the result equal to zero.

Risk Aversion and Portfolio Allocation

The general problem

$$\begin{aligned} \max_a \quad & \pi \ln[Y_0(1 + r_f) + a(r_G - r_f)] \\ & + (1 - \pi) \ln[Y_0(1 + r_f) + a(r_B - r_f)] \end{aligned}$$

has first-order condition

$$\frac{\pi(r_G - r_f)}{Y_0(1 + r_f) + a^*(r_G - r_f)} + \frac{(1 - \pi)(r_B - r_f)}{Y_0(1 + r_f) + a^*(r_B - r_f)} = 0.$$

Risk Aversion and Portfolio Allocation

$$\frac{\pi(r_G - r_f)}{Y_0(1 + r_f) + a^*(r_G - r_f)} + \frac{(1 - \pi)(r_B - r_f)}{Y_0(1 + r_f) + a^*(r_B - r_f)} = 0$$

$$\begin{aligned} & \pi(r_G - r_f)[Y_0(1 + r_f) + a^*(r_B - r_f)] \\ = & -(1 - \pi)(r_B - r_f)[Y_0(1 + r_f) + a^*(r_G - r_f)] \end{aligned}$$

$$\begin{aligned} & a^*(r_G - r_f)(r_B - r_f) \\ = & -Y_0(1 + r_f)[\pi(r_G - r_f) + (1 - \pi)(r_B - r_f)] \end{aligned}$$

Risk Aversion and Portfolio Allocation

$$\begin{aligned} & a^*(r_G - r_f)(r_B - r_f) \\ = & -Y_0(1 + r_f)[\pi(r_G - r_f) + (1 - \pi)(r_B - r_f)] \end{aligned}$$

implies

$$\frac{a^*}{Y_0} = -\frac{(1 + r_f)[\pi(r_G - r_f) + (1 - \pi)(r_B - r_f)]}{(r_G - r_f)(r_B - r_f)},$$

which is positive, since $r_G > r_f > r_B$ and

$$E(\tilde{r}) - r_f = \pi(r_G - r_f) + (1 - \pi)(r_B - r_f) > 0.$$

Risk Aversion and Portfolio Allocation

$$\frac{a^*}{Y_0} = - \frac{(1 + r_f)[\pi(r_G - r_f) + (1 - \pi)(r_B - r_f)]}{(r_G - r_f)(r_B - r_f)},$$

In this case, a^* :

Rises proportionally with Y_0 .

Increases as $E(\tilde{r}) - r_f$ rises.

Falls as r_G and r_B move farther away from r_f , holding $E(\tilde{r})$ constant; that is, in response to a “mean preserving spread.”

Risk Aversion and Portfolio Allocation

$$\frac{a^*}{Y_0} = - \frac{(1 + r_f)[\pi(r_G - r_f) + (1 - \pi)(r_B - r_f)]}{(r_G - r_f)(r_B - r_f)},$$

r_f	r_G	r_B	π	$E(\tilde{r})$	a^*/Y_0
0.05	0.40	-0.20	0.50	0.10	0.60
0.05	0.30	-0.10	0.50	0.10	1.40
0.05	0.40	-0.20	0.75	0.25	2.40

The fraction of initial wealth allocated to stocks rises when stocks become less risky or pay higher expected returns.

Portfolios, Risk Aversion, and Wealth

Let's generalize our previous example with logarithmic utility to the case where

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1-\gamma},$$

with $\gamma > 0$. For this Bernoulli utility function, the coefficient of relative risk aversion is constant and equal to γ . The specific setting $\gamma = 1$ takes us back to the case with logarithmic utility.

Portfolios, Risk Aversion, and Wealth

Hence, in this extended example,

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1-\gamma} \text{ implies } u'(Y) = Y^{-\gamma} = \frac{1}{Y^\gamma}.$$

and stock returns can either be good or bad

$$\tilde{r} = \begin{cases} r_G & \text{with probability } \pi \\ r_B & \text{with probability } 1 - \pi \end{cases}$$

where $r_G > r_f > r_B$ defines the “good” and “bad” states and

$$\pi r_G + (1 - \pi)r_B > r_f,$$

so that $E(\tilde{r}) > r_f$ and the investor will choose $a^* > 0$.

Portfolios, Risk Aversion, and Wealth

With CRRA (constant relative risk aversion) utility and two states for \tilde{r} , the problem

$$\max_a E\{u[Y_0(1 + r_f) + a(\tilde{r} - r_f)]\}$$

specializes to

$$\begin{aligned} \max_a \quad & \pi \left\{ \frac{[Y_0(1 + r_f) + a(r_G - r_f)]^{1-\gamma} - 1}{1 - \gamma} \right\} \\ & + (1 - \pi) \left\{ \frac{[Y_0(1 + r_f) + a(r_B - r_f)]^{1-\gamma} - 1}{1 - \gamma} \right\} \end{aligned}$$

Portfolios, Risk Aversion, and Wealth

The problem

$$\max_a \pi \left\{ \frac{[Y_0(1+r_f) + a(r_G - r_f)]^{1-\gamma} - 1}{1-\gamma} \right\} \\ + (1-\pi) \left\{ \frac{[Y_0(1+r_f) + a(r_B - r_f)]^{1-\gamma} - 1}{1-\gamma} \right\}$$

has first-order condition

$$\frac{\pi(r_G - r_f)}{[Y_0(1+r_f) + a^*(r_G - r_f)]^\gamma} + \frac{(1-\pi)(r_B - r_f)}{[Y_0(1+r_f) + a^*(r_B - r_f)]^\gamma} = 0.$$

Portfolios, Risk Aversion, and Wealth

$$\frac{\pi(r_G - r_f)}{[Y_0(1 + r_f) + a^*(r_G - r_f)]^\gamma} + \frac{(1 - \pi)(r_B - r_f)}{[Y_0(1 + r_f) + a^*(r_B - r_f)]^\gamma} = 0.$$

Notice that with $\gamma = 1$, this first-order condition specializes to the one that we derived last time, assuming log utility.

Portfolios, Risk Aversion, and Wealth

$$\frac{\pi(r_G - r_f)}{[Y_0(1 + r_f) + a^*(r_G - r_f)]^\gamma} + \frac{(1 - \pi)(r_B - r_f)}{[Y_0(1 + r_f) + a^*(r_B - r_f)]^\gamma} = 0.$$

Problem Set 10, Question 2: Set $Y_0 = 100$, $r_G = 0.30$, $r_B = 0.05$, $r_f = 0.10$, $\pi = 1 - \pi = 1/2$, and $\gamma = 2$. Since the constant coefficient of relative risk aversion goes up to 2 from 1, we should expect to see a^* decline, relative to question 1.

Portfolios, Risk Aversion, and Wealth

Problem Set 10, Question 2: $Y_0 = 100$, $r_G = 0.30$, $r_B = 0.05$, $r_f = 0.10$, $\pi = 1 - \pi = 1/2$, and $\gamma = 2$.

$$\frac{\pi(r_G - r_f)}{[Y_0(1 + r_f) + a^*(r_G - r_f)]^\gamma} + \frac{(1 - \pi)(r_B - r_f)}{[Y_0(1 + r_f) + a^*(r_B - r_f)]^\gamma} = 0.$$

$$\frac{(1/2)(0.20)}{(110 + 0.20a^*)^2} + \frac{(1/2)(-0.05)}{(110 - 0.05a^*)^2} = 0.$$

Portfolios, Risk Aversion, and Wealth

Problem Set 10, Question 2: $Y_0 = 100$, $r_G = 0.30$, $r_B = 0.05$, $r_f = 0.10$, $\pi = 1 - \pi = 1/2$, and $\gamma = 2$.

With $\gamma = 2$,

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma} = \frac{Y^{-1} - 1}{-1} = 1 - Y^{-1}$$

Portfolios, Risk Aversion, and Wealth

Problem Set 10, Question 2: $Y_0 = 100$, $r_G = 0.30$, $r_B = 0.05$,
 $r_f = 0.10$, $\pi = 1 - \pi = 1/2$, and $\gamma = 2$ so that
 $u(Y) = 1 - Y^{-1}$

$$\max_a (1/2)\{1 - [(1 + r_f)Y_0 + a(r_G - r_f)]^{-1}\} \\ + (1/2)\{1 - [(1 + r_f)Y_0 + a(r_B - r_f)]^{-1}\}$$

$$\max_a (1/2)[1 - (110 + 0.20a)^{-1}] + (1/2)[1 - (110 - 0.05a)^{-1}]$$

Portfolios, Risk Aversion, and Wealth

$$\max_a (1/2)[1 - (110 + 0.20a)^{-1}] + (1/2)[1 - (110 - 0.05a)^{-1}]$$

$$\frac{(1/2)(0.20)}{(110 + 0.20a^*)^2} + \frac{(1/2)(-0.05)}{(110 - 0.05a^*)^2} = 0$$

Portfolios, Risk Aversion, and Wealth

$$\frac{(1/2)(0.20)}{(110 + 0.20a^*)^2} + \frac{(1/2)(-0.05)}{(110 - 0.05a^*)^2} = 0.$$

$$\frac{(1/2)(0.20)}{(110 + 0.20a^*)^2} = \frac{(1/2)(0.05)}{(110 - 0.05a^*)^2}$$

$$4 = \frac{(110 + 0.20a^*)^2}{(110 - 0.05a^*)^2} = \left(\frac{110 + 0.20a^*}{110 - 0.05a^*} \right)^2$$

Portfolios, Risk Aversion, and Wealth

$$4 = \left(\frac{110 + 0.20a^*}{110 - 0.05a^*} \right)^2$$

$$2 = \frac{110 + 0.20a^*}{110 - 0.05a^*}$$

From here, you can solve for a^* and confirm that the amount allocated to stocks by the investor with $\gamma = 2$ is smaller than the amount allocated to stocks by the investor with $\gamma = 1$.

Portfolios, Risk Aversion, and Wealth

More generally,

$$\frac{\pi(r_G - r_f)}{[Y_0(1 + r_f) + a^*(r_G - r_f)]^\gamma} + \frac{(1 - \pi)(r_B - r_f)}{[Y_0(1 + r_f) + a^*(r_B - r_f)]^\gamma} = 0$$

$$\begin{aligned} & \pi(r_G - r_f)[Y_0(1 + r_f) + a^*(r_B - r_f)]^\gamma \\ = & (1 - \pi)(r_f - r_B)[Y_0(1 + r_f) + a^*(r_G - r_f)]^\gamma \end{aligned}$$

$$\begin{aligned} & [\pi(r_G - r_f)]^{1/\gamma} [Y_0(1 + r_f) + a^*(r_B - r_f)] \\ = & [(1 - \pi)(r_f - r_B)]^{1/\gamma} [Y_0(1 + r_f) + a^*(r_G - r_f)] \end{aligned}$$

Portfolios, Risk Aversion, and Wealth

$$\begin{aligned} & [\pi(r_G - r_f)]^{1/\gamma} [Y_0(1 + r_f) + a^*(r_B - r_f)] \\ = & [(1 - \pi)(r_f - r_B)]^{1/\gamma} [Y_0(1 + r_f) + a^*(r_G - r_f)] \\ & Y_0(1 + r_f)[\pi(r_G - r_f)]^{1/\gamma} + a^*(r_B - r_f)[\pi(r_G - r_f)]^{1/\gamma} \\ = & Y_0(1 + r_f)[(1 - \pi)(r_f - r_B)]^{1/\gamma} \\ & + a^*(r_G - r_f)[(1 - \pi)(r_f - r_B)]^{1/\gamma} \\ & Y_0(1 + r_f)\{[\pi(r_G - r_f)]^{1/\gamma} - [(1 - \pi)(r_f - r_B)]^{1/\gamma}\} \\ = & a^*\{(r_G - r_f)[(1 - \pi)(r_f - r_B)]^{1/\gamma} + (r_f - r_B)[\pi(r_G - r_f)]^{1/\gamma}\} \end{aligned}$$

Portfolios, Risk Aversion, and Wealth

$$\begin{aligned} & Y_0(1 + r_f)\{\pi(r_G - r_f)]^{1/\gamma} - [(1 - \pi)(r_f - r_B)]^{1/\gamma}\} \\ = & a^*\{(r_G - r_f)[(1 - \pi)(r_f - r_B)]^{1/\gamma} + (r_f - r_B)[\pi(r_G - r_f)]^{1/\gamma}\} \end{aligned}$$

implies

$$\frac{a^*}{Y_0} = \frac{(1 + r_f)\{\pi(r_G - r_f)]^{1/\gamma} - [(1 - \pi)(r_f - r_B)]^{1/\gamma}\}}{(r_G - r_f)[(1 - \pi)(r_f - r_B)]^{1/\gamma} + (r_f - r_B)[\pi(r_G - r_f)]^{1/\gamma}}$$

Portfolios, Risk Aversion, and Wealth

$$\frac{a^*}{Y_0} = \frac{(1 + r_f) \{ [\pi(r_G - r_f)]^{1/\gamma} - [(1 - \pi)(r_f - r_B)]^{1/\gamma} \}}{(r_G - r_f)[(1 - \pi)(r_f - r_B)]^{1/\gamma} + (r_f - r_B)[\pi(r_G - r_f)]^{1/\gamma}}$$

γ	r_f	r_G	r_B	π	$E(\tilde{r})$	a^*/Y_0
0.5	0.05	0.40	-0.20	0.50	0.10	1.20
1	0.05	0.40	-0.20	0.50	0.10	0.60
2	0.05	0.40	-0.20	0.50	0.10	0.30
3	0.05	0.40	-0.20	0.50	0.10	0.20
5	0.05	0.40	-0.20	0.50	0.10	0.12
10	0.05	0.40	-0.20	0.50	0.10	0.06

Portfolios, Risk Aversion, and Wealth

γ	r_f	r_G	r_B	π	$E(\tilde{r})$	a^*/Y_0
0.5	0.05	0.40	-0.20	0.50	0.10	1.20
1	0.05	0.40	-0.20	0.50	0.10	0.60
2	0.05	0.40	-0.20	0.50	0.10	0.30
3	0.05	0.40	-0.20	0.50	0.10	0.20
5	0.05	0.40	-0.20	0.50	0.10	0.12
10	0.05	0.40	-0.20	0.50	0.10	0.06

Consistent with Arrow's theorem, higher coefficients of relative risk aversion are associated with smaller values of a^* .

Portfolios, Risk Aversion, and Wealth

$$\frac{a^*}{Y_0} = \frac{(1 + r_f) \{ [\pi(r_G - r_f)]^{1/\gamma} - [(1 - \pi)(r_f - r_B)]^{1/\gamma} \}}{(r_G - r_f)[(1 - \pi)(r_f - r_B)]^{1/\gamma} + (r_f - r_B)[\pi(r_G - r_f)]^{1/\gamma}}$$

Also consistent with Arrow's results, we see here that with constant relative risk aversion, a^* rises proportionally with wealth.