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FINANCIAL ECONOMICS

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Pricing Safe Cash Flows

Consider an asset that generates an arbitrary stream of safe (riskless) cash flows $C_1, C_2, \ldots, C_T$, over the next T years.

To simplify the task of "pricing" this asset, we might view it as a portfolio of more basic assets: one that pays C_1 for sure in one year, one that pays C_2 for sure in two years, ..., and one that pays C_T for sure in T years.

The price of the multi-period asset must equal the sum of the prices of the more basic assets.

Pricing Safe Cash Flows

We've now reduced the problem of pricing any riskless asset to the simpler problem of pricing a more basic asset that pays C_t for sure t years from now.

But this more basic asset has the same payoff as C_t t-year discount bonds. Its price P_t^A today must equal

$$P_t^A = C_t P_t$$

where P_t is the price of a t-year discount bond.

Pricing Safe Cash Flows

And since the price and interest rate of the t-year discount bond are related via

$$P_t = \frac{1}{(1 + r_t)^t}$$

It is also the case that

$$P_t^A = C_t P_t = \frac{C_t}{(1 + r_t)^t}$$

the price of any safe asset equals the present discounted value of its future cash flows.

Pricing Safe Cash Flows

$$C_t P_t = P_t^A = \frac{C_t}{(1 + r_t)^t}$$

How can we extend these pricing principles to apply to risk assets? There are two possibilities, one in each direction.

Pricing Risky Cash Flows

Consider a risky asset, with cash flows $\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_T$ over the next T years that are random variables with values that are unknown today.

Again, we might simplify the task of pricing this asset, by viewing it as a portfolio of more basic assets, each of which makes a random payment $\tilde{C}_t$ after t years, then summing up the prices of all of these more basic assets.

But we still have to deal with the fact that the payoff $\tilde{C}_t$ is risky.

Pricing Risky Cash Flows

One possibility is to break down the random payoff $\tilde{C}_t$ into separate components $C_{t,1}, C_{t,2}, \ldots, C_{t,n}$ delivered in n different "states of the world" that can prevail t years from now.

The risky asset that delivers the random payoff $\tilde{C}_t$ t years from now can itself be viewed as a portfolio of contingent claims: $C_{t,1}$ contingent claims for state 1, $C_{t,2}$ contingent claims for state 2, $\ldots$, and $C_{t,n}$ contingent claims for state n.

Pricing Risky Cash Flows

This Arrow-Debreu approach to asset pricing then computes

$$P_t^A = q_{t,1}C_{t,1} + q_{t,2}C_{t,2} + \ldots + q_{t,n}C_{t,n}$$

where $q_{t,i}$ is the price today of a contingent claim that delivers one dollar if state i occurs t years from now and zero otherwise, inferred most likely from prices of options on the S&P 500.

This approach uses contingent claims as the "basic building blocks" for risky assets, in the same way that discount bonds can be viewed as the building blocks for coupon bonds.

Pricing Risky Cash Flows

In probability theory, if a random variable $\tilde{X}$ can take on n possible values, $X_1, X_2, \ldots, X_n$, with probabilities $\pi_1, \pi_2, \ldots, \pi_n$, then the expected value of $\tilde{X}$ is

$$E(\tilde{X}) = \pi_1 X_1 + \pi_2 X_2 + \ldots + \pi_n X_n.$$

Pricing Risky Cash Flows

More traditional approaches to asset pricing replace the random payoff $\tilde{C}_t$ with its expected value $E(\tilde{C}_t)$ and then "penalize" the fact that the payoff is random by discounting it at a higher rate

$$P_t^A = \frac{E(\tilde{C}_t)}{(1 + r_t + \psi_t)^t}$$

The capital asset pricing model (CAPM) will give us a way of determining values for the risk premium ψ_t.

Two Perspectives on Asset Pricing

$$\text{Arrow-Debreu} \Leftarrow C_t P_t = P_t^A = \frac{C_t}{(1+r_t)^t} \Rightarrow \text{CAPM}$$

Two Perspectives on Asset Pricing

Although all are designed to accomplish the same basic goal – to value risky cash flows – different theories of asset pricing can be grouped under two broad headings.

No-arbitrage theories take the prices of some assets as given and use those to determine the prices of other assets.

Equilibrium theories price all assets based on the principles of microeconomic theory.

Two Perspectives on Asset Pricing

No-arbitrage theories require fewer assumptions and are sometimes easier to use.

We've already used no-arbitrage arguments, for example, to price stocks and bonds as portfolios of contingent claims and to price coupon bonds as portfolios of discount bonds.

Two Perspectives on Asset Pricing

But no-arbitrage theories raise questions that only equilibrium theories can answer.

Where do the prices of the basic securities come from? And how do asset prices relate to economic fundamentals?

As an equilibrium theory of asset pricing, the CAPM will also help us answer these questions.

Two Perspectives on Asset Pricing

Whereas no-arbitrage theories require only that investors prefer more to less . . .

. . . the CAPM requires us to make assumptions about how investors trade off risk versus return.

Two Perspectives on Asset Pricing

	Arrow-Debreu	CAPM
Analytic Device	contingent claims	risk premia
Scope of Theory	no-arbitrage	equilibrium
Assumptions	more preferred to less	+ risk aversion

Preferences and Utility Functions

Consumers have preferences.

Economists describe those preferences with a utility function.

What exactly does this mean? And what exactly do economists assume when they describe or "represent" preferences with a utility function?

Preferences and Utility Functions

In the standard, static setting without uncertainty, let $c^1 = (c_a^1, c_b^1)$ and $c^2 = (c_a^2, c_b^2)$ denote two bundles of apples and bananas.

"More preferred to less" is enough to predict which bundle the consumer will choose if one of the bundles has more of both goods than the other.

Even if there is a trade-off, however, we should expect the consumer to be able to say which bundle is preferred or to express indifference.

Preferences and Utility Functions

Economists say that the consumer's preferences are represented by a utility function U when:

The consumer says "I prefer c^1 to c^2"

if and only if

$$U(c_a^1, c_b^1) > U(c_a^2, c_b^2)$$

Preferences and Utility Functions

Under what assumptions can preferences be represented by a utility function? Gerard Debreu answered this question in 1954:

Cowles Foundation Paper 97
Reprinted from Thrall, Davis and Coombs , eds.,
Decision Processes, John Wiley, 1954

CHAPTER XI

REPRESENTATION OF A PREFERENCE
ORDERING BY A NUMERICAL FUNCTION*

by

Gerard Debreu

COWLES COMMISSION FOR RESEARCH IN ECONOMICS

Theorem If preferences are complete, transitive, and continuous, then they can be represented by a continuous, real-valued utility function.

Preferences and Expected Utility Functions

Under certainty, the “goods” are described by consumption baskets with known characteristics.

Under uncertainty, the “goods” are random (state-contingent) payoffs.

Accordingly, let $p^1 = (p^1_G, p^1_B)$ and $p^2 = (p^2_G, p^2_B)$ denote the payoffs from two assets in a good state that occurs with probability π and a bad state with probability $1 - \pi$.

Preferences and Expected Utility Functions

In 1947, John von Neumann and Oskar Morgenstern worked out the conditions under which investors' preferences over risky payoffs could be described by an expected utility function such as

$$U(p) = E[u(p)] = \pi u(p_G) + (1-\pi)u(p_B),$$

where the Bernoulli utility function (named after Daniel Bernoulli, from the 1700s) over payoffs u is increasing and concave and the von Neumann-Morgenstern expected utility function U is linear in the probabilities.

Preferences and Expected Utility Functions

Economists say that an investor's preferences are represented by an expected utility function U when:

The investor says "I prefer p^1 to p^2"

if and only if

$$\begin{aligned} U(p_G^1, p_B^1) &= \pi u(p_G^1) + (1-\pi)u(p_B^1) \\ &> \pi u(p_G^2) + (1-\pi)u(p_B^2) \\ &= U(p_G^2, p_B^2) \end{aligned}$$

Preferences and Expected Utility Functions

Debreu's 1954 theorem says that if preferences over state-contingent payoffs (assets or "lotteries") are complete, transitive, and continuous, then they can be represented by some utility function.

von Neumann and Morgenstern identified the extra assumptions needed for that utility function to take the special form of an expected utility function.

Maurice Allais (Nobel Prize 1988) showed that these extra assumptions are quite "strong," in the sense that it is not difficult to construct examples where perfectly reasonable and rational investors make choices that violate those assumptions.

Preferences and Expected Utility Functions

Expected utility remains the dominant framework for analyzing economic decision-making under uncertainty.

But a very active line of ongoing research continues to explore alternatives and generalizations.

Preferences and Expected Utility Functions

For more detail on these issues, see:

Danthine and Donaldson, Ch 3 ("Making Choices in Risky Situations").

Mark Machina, "Choice Under Uncertainty: Problems Solved and Unsolved," *Journal of Economic Perspectives* (Summer 1987).