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FINANCIAL ECONOMICS

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Consumer Optimization: The Risk Dimension

Today, the consumer divides his or her income up into an amount to be consumed and amounts used to purchase the two contingent claims:

$$Y_0 \geq c_0 + q^G s^G + q^B s^B,$$

where s^G and s^B denote the number of each contingent claim purchased or sold short.

If either s^G or s^B is negative, the consumer is taking a short position in that claim.

Consumer Optimization: The Risk Dimension

Next year, the consumer simply spends his or her income, including payoffs on contingent claims:

$$Y_1^G + s^G \geq c_1^G$$

in the good state and

$$Y_1^B + s^B \geq c_1^B$$

in the bad state.

Consumer Optimization: The Risk Dimension

$$Y_0 \geq c_0 + q^G s^G + q^B s^B$$
$$Y_1^G + s^G \geq c_1^G$$
$$Y_1^B + s^B \geq c_1^B$$

Multiply both sides of the second equation by q^G and both sides of the third equation by q^B, Then add them all up to get the lifetime budget constraint

$$Y_0 + q^G Y_1^G + q^B Y_1^B \geq c_0 + q^G c_1^G + q^B c_1^B.$$

Consumer Optimization: The Risk Dimension

The problem is to choose c_0, c_1^G, and c_1^B to maximize expected utility

$$u(c_0) + \beta \pi u(c_1^G) + \beta(1-\pi) u(c_1^B),$$

subject to the budget constraint

$$Y_0 + q^G Y_1^G + q^B Y_1^B \geq c_0 + q^G c_1^G + q^B c_1^B.$$

This was Arrow and Debreu's key insight: that finance is like grocery shopping. Mathematically, making decisions over time and under uncertainty is no different from choosing apples, bananas, and pears!

Consumer Optimization: The Risk Dimension

The Lagrangian is

$$
\begin{aligned}
L &= u(c_0) + \beta\pi u(c_1^G) + \beta(1-\pi)u(c_1^B) \\
&\quad + \lambda \left(Y_0 + q^G Y_1^G + q^B Y_1^B - c_0 - q^G c_1^G - q^B c_1^B \right),
\end{aligned}
$$

and the first-order conditions are

$$u'(c_0^*) - \lambda^* = 0$$

$$\beta\pi u'(c_1^{G*}) - \lambda^* q^G = 0$$

$$\beta(1-\pi)u'(c_1^{B*}) - \lambda^* q^B = 0$$

Consumer Optimization: The Risk Dimension

The first-order conditions

$$u'(c_0^*) - \lambda^* = 0$$

$$\beta\pi u'(c_1^{G*}) - \lambda^* q^G = 0$$

$$\beta(1-\pi)u'(c_1^{B*}) - \lambda^* q^B = 0$$

imply that marginal rates of substitution equal relative prices:

$$\frac{u'(c_0^*)}{\beta\pi u'(c_1^{G*})} = \frac{1}{q^G} \text{ and } \frac{u'(c_0^*)}{\beta(1-\pi)u'(c_1^{B*})} = \frac{1}{q^B}$$

$$\text{and } \frac{\pi u'(c_1^{G*})}{(1-\pi)u'(c_1^{B*})} = \frac{q^G}{q^B}.$$