ECON 337901
FINANCIAL ECONOMICS

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Course Administration

Course webpage:

`http://irelandp.com/econ3379.html`

Notes/slides, handouts, problems sets and solutions, old exams.

Course Administration

Book:

Jean-Pierre Danthine and John Donaldson, *Intermediate Financial Theory*, Second (2005) or Third (2014) edition.

Recommended, but not required.

Third edition is freely available in electronic form via BC libraries.

Course Administration

Grading:

Homeworks: $1/3$

Take-Home Midterm: $1/3$

Take-Home Final: $1/3$

Course Administration

Academic integrity:

`https://www.bc.edu/content/bc-web/academics/sites/university-catalog/policies-procedures.html`

It's fine to work with other students on the homeworks, but please hand in our own answers for the purposes of grading.

Your work on the midterm and final should be yours and yours alone.

1 Mathematical and Economic Foundations

- A Mathematical Preliminaries
  - 1 Unconstrained Optimization
  - 2 Constrained Optimization
- B Consumer Optimization
  - 1 Graphical Analysis
  - 2 Algebraic Analysis
  - 3 The Time Dimension
  - 4 The Risk Dimension

Mathematical Preliminaries

Unconstrained Optimization

$$\max_x F(x)$$

Constrained Optimization

$$\max_x F(x) \text{ subject to } c \geq G(x)$$

Unconstrained Optimization

To find the value of x that solves

$$\max_x F(x)$$

you can:

1. Try out every possible value of x.
2. Use calculus.

Since search could take forever, let's use calculus instead.

Unconstrained Optimization

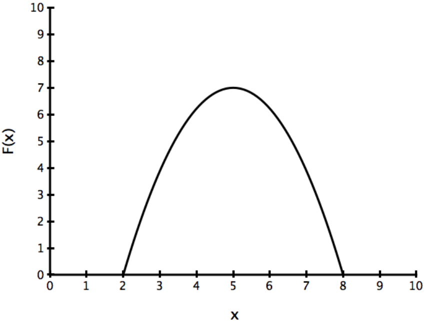
Theorem If x^* solves

$$\max_x F(x),$$

then x^* is a critical point of F, that is,

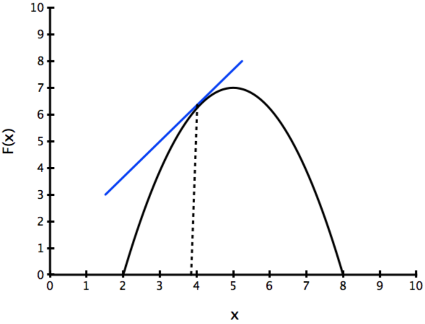
$$F'(x^*) = 0.$$

Unconstrained Optimization



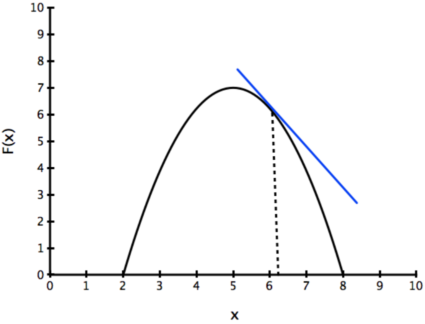
$F(x)$ maximized at $x^* = 5$

Unconstrained Optimization



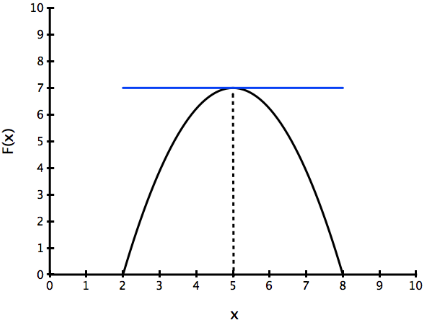
$F'(x) > 0$ when $x < 5$. $F(x)$ can be increased by increasing x.

Unconstrained Optimization



$F'(x) < 0$ when $x > 5$. $F(x)$ can be increased by decreasing x.

Unconstrained Optimization



$F'(x) = 0$ when $x = 5$. $F(x)$ is maximized.

Unconstrained Optimization

Theorem If x^* solves

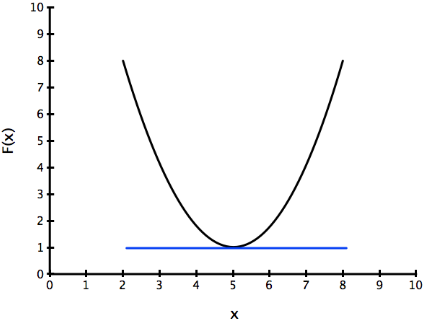
$$\max_x F(x),$$

then x^* is a critical point of F, that is,

$$F'(x^*) = 0.$$

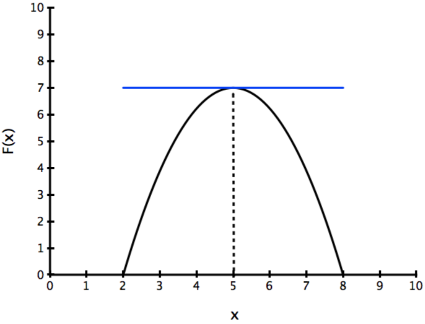
Note that the same first-order necessary condition $F'(x^*) = 0$ also characterizes a value of x^* that minimizes $F(x)$.

Unconstrained Optimization



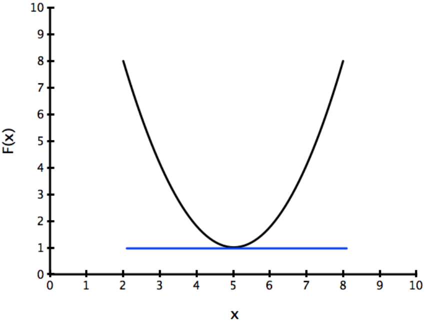
$F'(x) = 0$ when $x = 5$. $F(x)$ is minimized.

Unconstrained Optimization



$F'(x) = 0$ and $F''(x) < 0$ when $x = 5$. $F(x)$ is maximized.

Unconstrained Optimization



$F'(x) = 0$ and $F''(x) > 0$ when $x = 5$. $F(x)$ is minimized.

Unconstrained Optimization

Theorem If x^* solves

$$\max_x F(x),$$

then x^* is a critical point of F, that is,

$$F'(x^*) = 0.$$

Unconstrained Optimization

Theorem If

$$F'(x^*) = 0 \text{ and } F''(x^*) < 0,$$

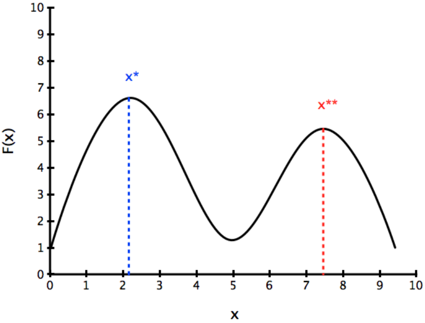
then x^* solves

$$\max_x F(x)$$

(at least locally).

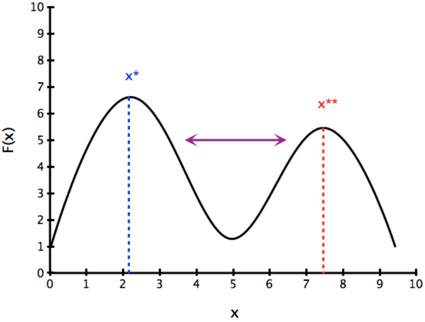
The first-order condition $F'(x^*) = 0$ and the second-order condition $F''(x^*) < 0$ are sufficient conditions for the value of x that (locally) maximizes $F(x)$.

Unconstrained Optimization



$F'(x^{**}) = 0$ and $F''(x^{**}) < 0$ at the local maximizer x^{**} and
$F'(x^{*}) = 0$ and $F''(x^{*}) < 0$ at the global maximizer x^{*}.

Unconstrained Optimization



$F'(x^{**}) = 0$ and $F''(x^{**}) < 0$ at the local maximizer x^{**} and $F'(x^*) = 0$ and $F''(x^*) < 0$ at the global maximizer x^*, but $F''(x) > 0$ in between x^* and x^{**}.

Unconstrained Optimization

Theorem If

$$F'(x^*) = 0$$

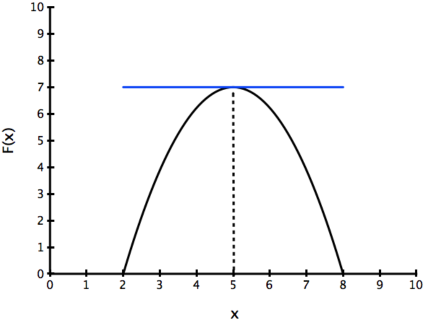
and

$$F''(x) < 0 \text{ for all } x \in \mathbb{R},$$

then x^* solves

$$\max_x F(x).$$

Unconstrained Optimization



$F''(x) < 0$ for all $x \in \mathbb{R}$ and $F'(5) = 0$. $F(x)$ is maximized when $x = 5$.

Unconstrained Optimization

If $F''(x) < 0$ for all $x \in \mathbb{R}$, then the function F is concave.

When F is concave, the first-order condition $F'(x^*) = 0$ is both necessary and sufficient for the value of x that maximizes $F(x)$.

And, as we are about to see, concave functions arise frequently and naturally in economics and finance.

Unconstrained Optimization

Recall that if $f(x) = ax$, then $f'(x) = a$.

If $g(x) = ax + b$, then $g'(x) = a$.

And if $h(x) = x^c$, then $h'(x) = cx^{c-1}$.

Unconstrained Optimization

Recall also the “chain rule”

If
$$h(x) = f(g(x)),$$
then
$$h'(x) = f'(g(x))g'(x)$$

Unconstrained Optimization

Example:

$$f(y) = -\left(\frac{1}{2}\right) y^2$$

$$g(x) = 2 - x$$

Then

$$h(x) = f(g(x)) = -\left(\frac{1}{2}\right)(2-x)^2$$

and

$$h'(x) = -\left(\frac{1}{2}\right) 2(2-x)^{2-1}(-1) = 2 - x$$

Unconstrained Optimization: Example 1

Consider the problem

$$\max_x \left(-\frac{1}{2}\right)(x-\tau)^2,$$

where τ is a number ($\tau \in \mathbb{R}$) that we might call the "target."

The first-order condition

$$-(x^* - \tau) = 0$$

leads us immediately to the solution: $x^* = \tau$.