ECON 337901
FINANCIAL ECONOMICS

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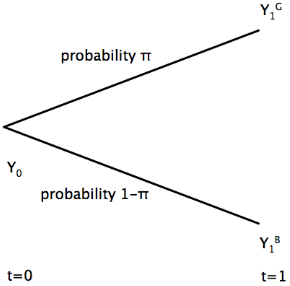
Consumer Optimization: The Risk Dimension



An event tree highlights randomness in income as the source of risk.

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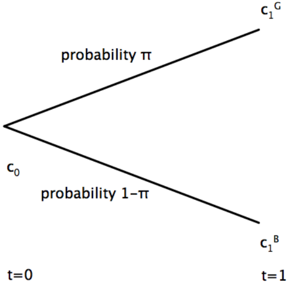

Uncertainty about future income "induces" randomness in future consumption as well.

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Suppose that the consumer's utility function is

$$u(c_0) + \beta\pi u(c_1^G) + \beta(1-\pi)u(c_1^B),$$

so that the terms involving next year's consumption are weighted by the probability that each state will occur as well as by the discount factor β.

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In probability theory, if a random variable X can take on n possible values, $X_1, X_2, \ldots, X_n$, with probabilities $\pi_1, \pi_2, \ldots, \pi_n$, then the expected value of X is

$$E(X) = \pi_1 X_1 + \pi_2 X_2 + \ldots + \pi_n X_n.$$

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Hence, by assuming that the consumer's utility function is

$$u(c_0) + \beta \pi u(c_1^G) + \beta(1 - \pi)u(c_1^B),$$

we are assuming that the consumer's seeks to maximize expected utility

$$u(c_0) + \beta E[u(c_1)].$$

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But by writing out all three terms,

$$u(c_0) + \beta\pi u(c_1^G) + \beta(1-\pi)u(c_1^B),$$

we can see that concavity of the function u, which in the standard microeconomic case represents a preference for diversity, represents here a preference for smoothness in consumption over time and across states in the future – the consumer is risk averse in the sense that he or she does not want consumption in the bad state to be too much different from consumption in the good state.

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To implement these state-contingent consumption plans, Arrow and Debreu imagined that the consumer would trade contingent claims for both future states.

A contingent claim for the good state costs q^G today, and delivers one unit of consumption next year in the good state and zero units of consumption next year in the bad state.

A contingent claim for the bad state costs q^B today, and delivers one unit of consumption next year in the bad state and zero units of consumption next year in the good state.

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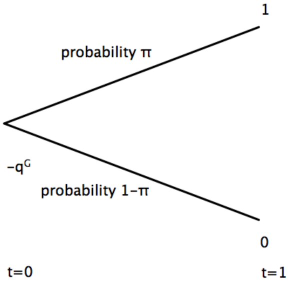

Payoffs for the contingent claim for the good state (a long position).

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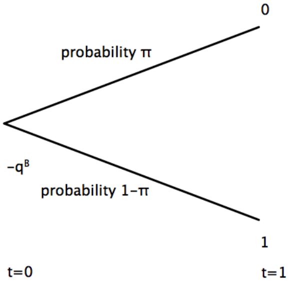

Payoffs for the contingent claim for the bad state (a long position).

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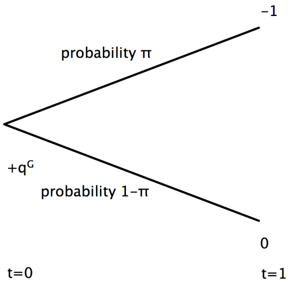

Payoffs for a short position in the contingent claim for the good state.

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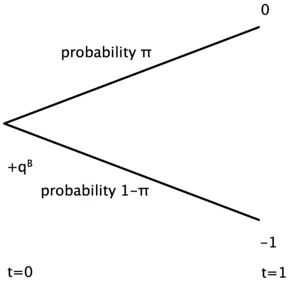

Payoffs for a short position in the contingent claim for the bad state.

Consumer Optimization: The Risk Dimension

Trading Strategy	Claim	Cash Flow at $t = 0$	Cash Flow in Good State at $t = 1$	Cash Flow in Bad State at $t = 1$
Long	Good	$-q^G$	$+1$	0
Long	Bad	$-q^B$	0	$+1$
Short	Good	$+q^G$	-1	0
Short	Bad	$+q^B$	0	-1

Like a sophisticated form of saving and borrowing, where the investor can “fine-tune” the future state in which payments are received or made.

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Today, the consumer divides his or her income up into an amount to be consumed and amounts used to purchase the two contingent claims:

$$Y_0 \geq c_0 + q^G s^G + q^B s^B,$$

where s^G and s^B denote the number of each contingent claim purchased or sold short.

If either s^G or s^B is negative, the consumer is taking a short position in that claim.

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Next year, the consumer simply spends his or her income, including payoffs on contingent claims:

$$Y_1^G + s^G \geq c_1^G$$

in the good state and

$$Y_1^B + s^B \geq c_1^B$$

in the bad state.

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$$Y_0 \geq c_0 + q^G s^G + q^B s^B$$
$$Y_1^G + s^G \geq c_1^G$$
$$Y_1^B + s^B \geq c_1^B$$

Multiply both sides of the second equation by q^G and both sides of the third equation by q^B, Then add them all up to get the lifetime budget constraint

$$Y_0 + q^G Y_1^G + q^B Y_1^B \geq c_0 + q^G c_1^G + q^B c_1^B.$$

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The problem is to choose c_0, c_1^G, and c_1^B to maximize expected utility

$$u(c_0) + \beta\pi u(c_1^G) + \beta(1-\pi)u(c_1^B),$$

subject to the budget constraint

$$Y_0 + q^G Y_1^G + q^B Y_1^B \geq c_0 + q^G c_1^G + q^B c_1^B.$$

This was Arrow and Debreu's key insight: that finance is like grocery shopping. Mathematically, making decisions over time and under uncertainty is no different from choosing apples, bananas, and pears!