ECON 337901
FINANCIAL ECONOMICS

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Constrained Optimization

A method for solving constrained optimization problems like

$$\max_x F(x) \text{ subject to } c \geq G(x)$$

was developed by Joseph-Louis Lagrange (France/Italy, 1736-1813) and extended by Harold Kuhn (US, 1925-2014) and Albert Tucker (US, 1905-1995).

Constrained Optimization

Associated with the problem:

$$\max_{x} F(x) \text{ subject to } c \geq G(x)$$

Define the Lagrangian

$$L(x, \lambda) = F(x) + \lambda[c - G(x)],$$

where λ is the Lagrange multiplier.

Constrained Optimization

Then, look for a critical point of the full Lagrangian

$$L(x, \lambda) = F(x) + \lambda[c - G(x)],$$

instead of just the objective function F by itself.

That is, use the FOC

$$F'(x^*) - \lambda^* G'(x^*) = 0.$$

Constrained Optimization

$$L(x, \lambda) = F(x) + \lambda[c - G(x)],$$

Theorem (Kuhn-Tucker) If x^* maximizes $F(x)$ subject to $c \geq G(x)$, then there exists a value $\lambda^* \geq 0$ such that, together, x^* and λ^* satisfy the first-order condition

$$F'(x^*) - \lambda^* G'(x^*) = 0$$

and the complementary slackness condition

$$\lambda^*[c - G(x^*)] = 0.$$

Constrained Optimization

In the case where $c > G(x^*)$, the constraint is non-binding.
The complementary slackness condition

$$\lambda^*[c - G(x^*)] = 0$$

requires that $\lambda^* = 0$.

And the first-order condition

$$F'(x^*) - \lambda^* G'(x^*) = 0$$

requires that $F'(x^*) = 0$.

Constrained Optimization

In the case where $c = G(x^*)$, the constraint is binding. The complementary slackness condition

$$\lambda^*[c - G(x^*)] = 0$$

puts no further restriction on $\lambda^* \geq 0$.

Now the first-order condition

$$F'(x^*) - \lambda^* G'(x^*) = 0$$

requires that $F'(x^*) = \lambda^* G'(x^*)$.

Constrained Optimization: Example 1

For the problem

$$\max_x \left(-\frac{1}{2}\right)(x-5)^2 \text{ subject to } 7 \geq x,$$

$F(x) = (-1/2)(x-5)^2$, $c = 7$, and $G(x) = x$. The Lagrangian is

$$L(x, \lambda) = \left(-\frac{1}{2}\right)(x-5)^2 + \lambda(7-x).$$

Constrained Optimization: Example 1

With

$$L(x, \lambda) = \left(-\frac{1}{2}\right)(x-5)^2 + \lambda(7-x),$$

the first-order condition

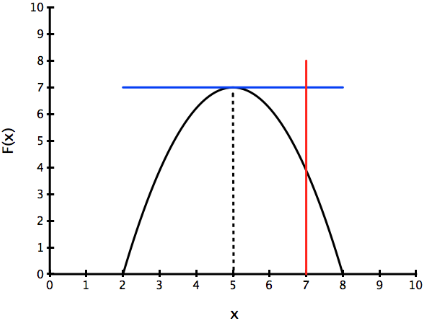
$$-(x^* - 5) - \lambda^* = 0$$

and the complementary slackness condition

$$\lambda^*(7 - x^*) = 0$$

are satisfied with $x^* = 5$, $F'(x^*) = 0$, $\lambda^* = 0$, and $7 > x^*$.

Constrained Optimization: Example 1



Here, the solution has $F'(x^*) = 0$ since the constraint is nonbinding.

Constrained Optimization: Example 2

For the problem

$$\max_x \left(-\frac{1}{2}\right)(x-5)^2 \text{ subject to } 4 \geq x,$$

$F(x) = (-1/2)(x-5)^2$, $c = 4$, and $G(x) = x$. The Lagrangian is

$$L(x,\lambda) = \left(-\frac{1}{2}\right)(x-5)^2 + \lambda(4-x).$$

Constrained Optimization: Example 2

With

$$L(x, \lambda) = \left(-\frac{1}{2}\right)(x-5)^2 + \lambda(4-x),$$

the first-order condition

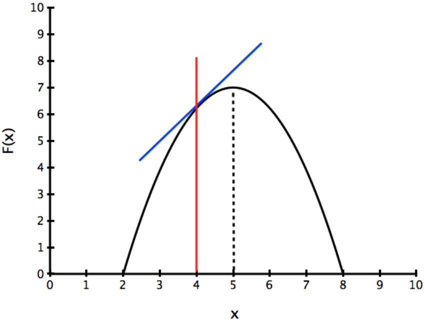
$$-(x^* - 5) - \lambda^* = 0$$

and the complementary slackness condition

$$\lambda^*(4 - x^*) = 0$$

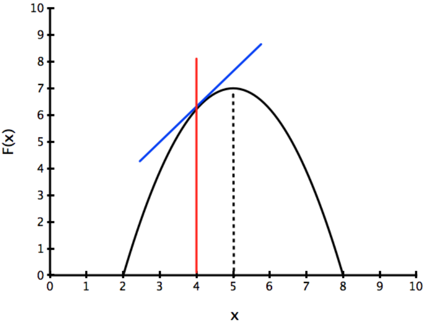
are satisfied with $x^* = 4$ and $F'(x^*) = \lambda^* = 1 > 0$.

Constrained Optimization: Example 2



Here, the solution has $F'(x^*) = \lambda^* G'(x^*) > 0$ since the constraint is binding. $F'(x^*) > 0$ indicates that we'd like to increase the value of x, but the constraint won't let us.

Constrained Optimization: Example 2



With a binding constraint, $F'(x^*) \neq 0$ but $F'(x^*) - \lambda^* G'(x^*) = 0$. The value x^* that solves the problem is a critical point, not of the objective function $F(x)$, but instead of the entire Lagrangian $F(x) + \lambda[c - G(x)]$.

Consumer Optimization

Alfred Marshall, *Principles of Economics*, 1890. – supply and demand

Francis Edgeworth, *Mathematical Psychics*, 1881.

Vilfredo Pareto, *Manual of Political Economy*, 1906. – indifference curves

Consumer Optimization

John Hicks, *Value and Capital*, 1939. – wealth and substitution effects

Paul Samuelson, *Foundations of Economic Analysis*, 1947. – mathematical reformulation

Irving Fisher, *The Theory of Interest*, 1930. – intertemporal extension.

Consumer Optimization

Gerard Debreu, *Theory of Value*, 1959.

Kenneth Arrow, “The Role of Securities in the Optimal Allocation of Risk Bearing,” *Review of Economic Studies*, 1964.

Extensions to include risk and uncertainty.

Problem Set 2

Problem set 2 presents an economic example that will give you practice solving a constrained optimization problem.

To make the problem more interesting, there will be two choice variables instead of just one.

And the results will give us a preview of consumer theory, which will be our focus next.

Problem Set 2

Y = income

c_a, c_b = consumption of apples and bananas

p_a, p_b = prices of apples and bananas

Problem Set 2

Budget constraint

$$Y \geq p_a c_a + p_b c_b$$

Utility:

$$\alpha \ln(c_a) + (1 - \alpha) \ln(c_b)$$

$0 \leq \alpha \leq 1$ is the weight on apples relative to bananas in the consumer's preferences

Problem Set 2

The problem:

$$\max_{c_a, c_b} \; \alpha \ln(c_a) + (1-\alpha)\ln(c_b) \text{ subject to } Y \geq p_a c_a + p_b c_b$$

The Lagrangian:

$$L(c_a, c_b, \lambda) = \alpha \ln(c_a) + (1-\alpha)\ln(c_b) + \lambda(Y - p_a c_a - p_b c_b)$$

Problem Set 2

The Lagrangian:

$$L(c_a, c_b, \lambda) = \alpha \ln(c_a) + (1-\alpha)\ln(c_b) + \lambda(Y - p_a c_a - p_b c_b)$$

With two choice variables there will be two first-order conditions, each derived in the same way: by differentiating the Lagrangian by one choice variable, holding the other constant.

Problem Set 2

$$L(c_a, c_b, \lambda) = \alpha \ln(c_a) + (1-\alpha)\ln(c_b) + \lambda(Y - p_a c_a - p_b c_b)$$

FOC for c_a:

$$\frac{\alpha}{c_a^*} - \lambda^* p_a = 0$$

FOC for c_b:

$$\frac{1-\alpha}{c_b^*} - \lambda^* p_b = 0$$

Problem Set 2

The FOCs

$$\frac{\alpha}{c_a^*} - \lambda^* p_a = 0$$

$$\frac{1-\alpha}{c_b^*} - \lambda^* p_b = 0$$

form a system of 2 equations in 3 unknowns: c_a^*, c_b^*, and λ^*.

Since the utility function $\alpha \ln(c_a) + (1-\alpha)\ln(c_b)$ implies that "more is preferred to less," we know in advance that the budget constraint will bind, providing a third equation

$$Y = p_a c_a^* + p_b c_b^*.$$

Problem Set 2

In general, there are many ways to solve the three-equation system

$$\frac{\alpha}{c_a^*} - \lambda^* p_a = 0$$

$$\frac{1-\alpha}{c_b^*} - \lambda^* p_b = 0$$

$$Y = p_a c_a^* + p_b c_b^*.$$

All will lead to the same solution!

Problem Set 2

One approach is to use the FOCs to solve for c_a^* and c_b^* in terms of λ^*:

$$\frac{\alpha}{c_a^*} - \lambda^* p_a = 0 \Rightarrow c_a^* = \frac{\alpha}{\lambda^* p_a}$$

$$\frac{1-\alpha}{c_b^*} - \lambda^* p_b = 0 \Rightarrow c_b^* = \frac{1-\alpha}{\lambda^* p_b}$$

Then substitute these expressions for c_a^* and c_b^* into the binding constraint

$$Y = p_a c_a^* + p_b c_b^*.$$

Problem Set 2

If you do this, you'll find that λ^* depends only on Y!

Then take your solution for λ^* and substitute it back into the two FOCs

$$c_a^* = \frac{\alpha}{\lambda^* p_a}$$

$$c_b^* = \frac{1-\alpha}{\lambda^* p_b}$$

Problem Set 2

Now you'll find that c_a^* depends on p_a, Y, and α.

And c_b^* depends on p_b, Y, and $1-\alpha$.

These solutions for c_a^* and c_b^* are the (Marshallian) demand curves for apples and bananas.

Problem Set 2

Now you'll find that c_a^* depends on p_a, Y, and α.

And c_b^* depends on p_b, Y, and $1 - \alpha$.

Both goods are "ordinary" goods (not Giffen goods): when the price goes up, quantity demanded goes down.

Problem Set 2

Now you'll find that c_a^* depends on p_a, Y, and α.

And c_b^* depends on p_b, Y, and $1 - \alpha$.

Both goods are "normal" goods: when the income goes up, quantity demanded goes up.

Problem Set 2

Finally, use your solutions to see how

$$\frac{p_a c_a^*}{Y} = \text{ share of income spent on apples}$$

depends on α.

And

$$\frac{p_b c_b^*}{Y} = \text{ share of income spent on bananas}$$

depends on $1 - \alpha$.