

# ECON 337901

# FINANCIAL ECONOMICS

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## Unconstrained Optimization: Example 2

Consider maximizing a function of three variables:

$$\max_{x_1, x_2, x_3} F(x_1, x_2, x_3)$$

Even if each variable can take on only 1,000 values, there are one billion possible combinations of  $(x_1, x_2, x_3)$  to search over!

This is an example of what Richard Bellman (US, 1920-1984) called the “curse of dimensionality.”

## Unconstrained Optimization: Example 2

Consider the problem:

$$\max_{x_1, x_2, x_3} \left(-\frac{1}{2}\right) (x_1 - \tau)^2 + \left(-\frac{1}{2}\right) (x_2 - x_1)^2 + \left(-\frac{1}{2}\right) (x_3 - x_2)^2.$$

Now the three first-order conditions

$$-(x_1^* - \tau) + (x_2^* - x_1^*) = 0$$

$$-(x_2^* - x_1^*) + (x_3^* - x_2^*) = 0$$

$$-(x_3^* - x_2^*) = 0$$

lead us to the solution:  $x_1^* = x_2^* = x_3^* = \tau$ .

# Constrained Optimization

To find the value of  $x$  that solves

$$\max_x F(x) \text{ subject to } c \geq G(x)$$

you can:

1. Try out every possible value of  $x$ .
2. Use calculus.

Since search could take forever, let's use calculus instead.

# Constrained Optimization

A method for solving constrained optimization problems like

$$\max_x F(x) \text{ subject to } c \geq G(x)$$

was developed by Joseph-Louis Lagrange (France/Italy, 1736-1813) and extended by Harold Kuhn (US, 1925-2014) and Albert Tucker (US, 1905-1995).

# Constrained Optimization

Associated with the problem:

$$\max_x F(x) \text{ subject to } c \geq G(x)$$

Define the **Lagrangian**

$$L(x, \lambda) = F(x) + \lambda[c - G(x)],$$

where  $\lambda$  is the **Lagrange multiplier**.

## Constrained Optimization

Then, look for a critical point of the full Lagrangian

$$L(x, \lambda) = F(x) + \lambda[c - G(x)],$$

instead of just the objective function  $F$  by itself.

That is, use the FOC

$$F'(x^*) - \lambda^* G'(x^*) = 0.$$

## Constrained Optimization

$$L(x, \lambda) = F(x) + \lambda[c - G(x)],$$

**Theorem (Kuhn-Tucker)** If  $x^*$  maximizes  $F(x)$  subject to  $c \geq G(x)$ , then there exists a value  $\lambda^* \geq 0$  such that, together,  $x^*$  and  $\lambda^*$  satisfy the **first-order condition**

$$F'(x^*) - \lambda^* G'(x^*) = 0$$

and the **complementary slackness condition**

$$\lambda^*[c - G(x^*)] = 0.$$



## Constrained Optimization

In the case where  $c > G(x^*)$ , the constraint is **non-binding**.  
The complementary slackness condition

$$\lambda^*[c - G(x^*)] = 0$$

requires that  $\lambda^* = 0$ .

And the first-order condition

$$F'(x^*) - \lambda^* G'(x^*) = 0$$

requires that  $F'(x^*) = 0$ .

## Constrained Optimization

In the case where  $c = G(x^*)$ , the constraint is **binding**. The complementary slackness condition

$$\lambda^*[c - G(x^*)] = 0$$

puts no further restriction on  $\lambda^* \geq 0$ .

Now the first-order condition

$$F'(x^*) - \lambda^* G'(x^*) = 0$$

requires that  $F'(x^*) = \lambda^* G'(x^*)$ .

## Constrained Optimization: Example 1

For the problem

$$\max_x \left( -\frac{1}{2} \right) (x - 5)^2 \text{ subject to } 7 \geq x,$$

$F(x) = (-1/2)(x - 5)^2$ ,  $c = 7$ , and  $G(x) = x$ . The Lagrangian is

$$L(x, \lambda) = \left( -\frac{1}{2} \right) (x - 5)^2 + \lambda(7 - x).$$

## Constrained Optimization: Example 1

With

$$L(x, \lambda) = \left(-\frac{1}{2}\right) (x - 5)^2 + \lambda(7 - x),$$

the first-order condition

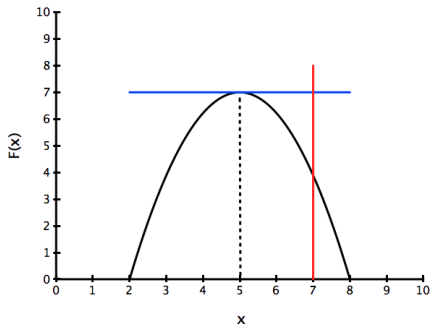
$$-(x^* - 5) - \lambda^* = 0$$

and the complementary slackness condition

$$\lambda^*(7 - x^*) = 0$$

are satisfied with  $x^* = 5$ ,  $F'(x^*) = 0$ ,  $\lambda^* = 0$ , and  $7 > x^*$ .

# Constrained Optimization: Example 1



Here, the solution has  $F'(x^*) = 0$  since the constraint is nonbinding.

## Constrained Optimization: Example 2

For the problem

$$\max_x \left( -\frac{1}{2} \right) (x - 5)^2 \text{ subject to } 4 \geq x,$$

$F(x) = (-1/2)(x - 5)^2$ ,  $c = 4$ , and  $G(x) = x$ . The Lagrangian is

$$L(x, \lambda) = \left( -\frac{1}{2} \right) (x - 5)^2 + \lambda(4 - x).$$

## Constrained Optimization: Example 2

With

$$L(x, \lambda) = \left(-\frac{1}{2}\right) (x - 5)^2 + \lambda(4 - x),$$

the first-order condition

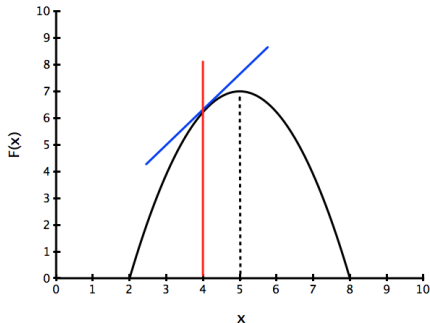
$$-(x^* - 5) - \lambda^* = 0$$

and the complementary slackness condition

$$\lambda^*(4 - x^*) = 0$$

are satisfied with  $x^* = 4$  and  $F'(x^*) = \lambda^* = 1 > 0$ .

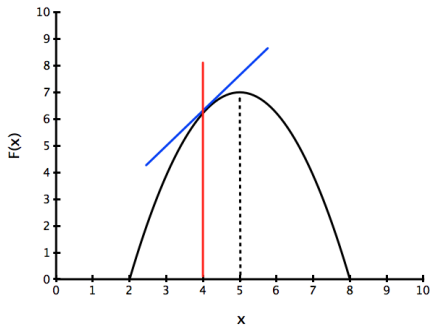
## Constrained Optimization: Example 2



Here, the solution has  $F'(x^*) = \lambda^* G'(x^*) > 0$  since the constraint is binding.  $F'(x^*) > 0$  indicates that we'd like to increase the value of  $x$ , but the constraint won't let us.



## Constrained Optimization: Example 2



With a binding constraint,  $F'(x^*) \neq 0$  but  $F'(x^*) - \lambda^* G'(x^*) = 0$ . The value  $x^*$  that solves the problem is a critical point, not of the objective function  $F(x)$ , but instead of the entire Lagrangian  $F(x) + \lambda[c - G(x)]$ .