ECON 337901
FINANCIAL ECONOMICS

Peter Ireland

Boston College

April 5, 2022

Generalizing the Portfolio Problem

With

$$\tilde{Y}_1 = (1 + r_f)Y_0 + \sum_{i=1}^{N} w_i Y_0(\tilde{r}_i - r_f),$$

the generalized problem can be stated as

$$\max_{w_1, w_2, \ldots, w_N} E\left\{u\left[Y_0(1 + r_f) + \sum_{i=1}^{N} w_i Y_0(\tilde{r}_i - r_f)\right]\right\}$$

Generalizing the Portfolio Problem

Modern Portfolio Theory examines the solution to this extended problem assuming that investors have mean-variance utility, that is, assuming that investors' preferences can be represented by a trade-off between the mean (expected value) and variance (or standard deviation) of terminal wealth.

MPT was developed by Harry Markowitz (US, b.1927, Nobel Prize 1990) in the early 1950s, the classic paper being his article "Portfolio Selection," *Journal of Finance* Vol.7 (March 1952): pp.77-91.

Justifying Mean-Variance Utility

The mean-variance utility hypothesis seemed natural at the time the MPT first appeared, and it retains some intuitive appeal today. But viewed in the context of more recent developments in financial economics, particularly the development of vN-M expected utility theory, it now looks a bit peculiar.

A first question for us, therefore, is: Under what conditions will investors have preferences over the means and variances of asset returns?

Justifying Mean-Variance Utility

If all individual risky asset returns are normally distributed, then terminal wealth $\tilde{Y}_1$ will be normally distributed as well.

And if $\tilde{Y}_1$ is normally distributed with mean $\mu_Y = E(\tilde{Y}_1)$ and standard deviation $\sigma_Y = \{E[\tilde{Y}_1 - E(\tilde{Y}_1)]^2\}^{1/2}$ then the expectation of any function of $\tilde{Y}_1$ can be written as a function of μ_Y and σ_Y:

$$E[u(\tilde{Y}_1)] = v(\mu_Y, \sigma_Y)$$

Justifying Mean-Variance Utility

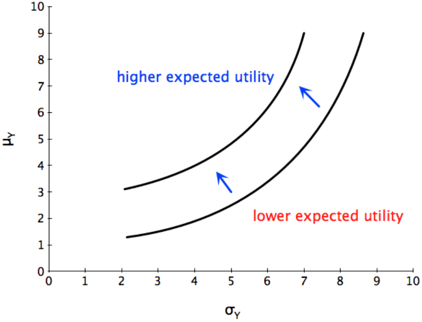
If $\tilde{Y}_1$ is normally distributed, there exists a function v such that

$$E[u(\tilde{Y}_1)] = v(\mu_Y, \sigma_Y).$$

Moreover, if $\tilde{Y}_1$ is normally distributed and

1. u is increasing, then v is increasing in μ_Y
2. u is concave, then v is decreasing in σ_Y
3. u is concave, then indifference curves defined over μ_Y and σ_Y are convex

Justifying Mean-Variance Utility



Since μ_Y is a "good" and σ_Y is a "bad," indifference curves slope up. But if u is concave, these indifference curves will still be convex.

Justifying Mean-Variance Utility

Returns on individual stocks and stock indices are approximately normal, but:

1. Returns on assets like options are highly non-normal.
2. Departures from normality, including skewness (asymmetry) and excess kurtosis ("fat tails"), can be detected in returns on individual stocks and the market as a whole.

Basically, stock market crashes happen more often than they would if returns were truly normal.

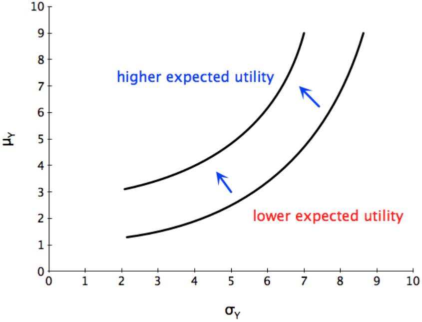
Justifying Mean-Variance Utility

The mean-variance utility hypothesis is intuitively appealing and can be justified with reference to vN-M expected utility theory by assuming risky asset returns are normally distributed.

That's why people say, "the CAPM requires normal returns."

It's also why people say, "the CAPM can't be used to price options."

Justifying Mean-Variance Utility



But what does the “budget constraint” look like in this diagram? To see, we need to consider the gains from diversification.

The Gains From Diversification

One of the most important lessons that we can take from modern portfolio theory involves the gains from diversification.

To see where these gains come from, consider forming a portfolio from two risky assets:

$$\tilde{r}_1, \tilde{r}_2 = \text{random returns}$$

$$\mu_1, \mu_2 = \text{expected returns}$$

$$\sigma_1, \sigma_2 = \text{standard deviations}$$

Assume $\mu_1 > \mu_2$ and $\sigma_1 > \sigma_2$ to create a trade-off between expected return and risk if the investor must choose between one or the other.

The Gains From Diversification

If w is the fraction of initial wealth allocated to asset 1 and $1 - w$ is the fraction of initial wealth allocated to asset 2, the random return $\tilde{r}_P$ on the portfolio is

$$\tilde{r}_P = w\tilde{r}_1 + (1 - w)\tilde{r}_2$$

and the expected return μ_P on the portfolio is

$$\begin{aligned} \mu_P &= E[w\tilde{r}_1 + (1 - w)\tilde{r}_2] \\ &= wE(\tilde{r}_1) + (1 - w)E(\tilde{r}_2) \\ &= w\mu_1 + (1 - w)\mu_2 \end{aligned}$$

The Gains From Diversification

$$\mu_P = w\mu_1 + (1 - w)\mu_2$$

The expected return on the portfolio is a weighted average of the expected returns on the individual assets.

Since $\mu_1 > \mu_2$, μ_P can range from μ_2 up to μ_1 as w increases from zero to one. Even higher (or lower) expected returns are possible if short selling is allowed.

The Gains From Diversification

But now let's calculate the variance of the random portfolio return

$$\tilde{r}_P = w\tilde{r}_1 + (1-w)\tilde{r}_2$$

$$\begin{aligned}
\sigma_P^2 &= E[(\tilde{r}_P - \mu_P)^2] \\
&= E\{[w\tilde{r}_1 + (1-w)\tilde{r}_2 - w\mu_1 - (1-w)\mu_2]^2\} \\
&= E\{[w(\tilde{r}_1 - \mu_1) + (1-w)(\tilde{r}_2 - \mu_2)]^2\} \\
&= E[w^2(\tilde{r}_1 - \mu_1)^2 + (1-w)^2(\tilde{r}_2 - \mu_2)^2 \\
&\quad + 2w(1-w)(\tilde{r}_1 - \mu_1)(\tilde{r}_2 - \mu_2)]
\end{aligned}$$

The Gains From Diversification

$$\begin{aligned}\sigma_P^2 &= E[w^2(\tilde{r}_1-\mu_1)^2+(1-w)^2(\tilde{r}_2-\mu_2)^2 \\ &\quad + 2w(1-w)(\tilde{r}_1-\mu_1)(\tilde{r}_2-\mu_2)]\end{aligned}$$

$$\begin{aligned}\sigma_P^2 &= w^2E[(\tilde{r}_1-\mu_1)^2]+(1-w)^2E[(\tilde{r}_2-\mu_2)^2] \\ &\quad + 2w(1-w)E[(\tilde{r}_1-\mu_1)(\tilde{r}_2-\mu_2)]\end{aligned}$$

The Gains From Diversification

In probability theory, the covariance between two random variables X_1 and X_2 is defined as

$$\sigma(X_1, X_2) = E\{[X_1 - E(X_1)][X_2 - E(X_2)]\}$$

and the correlation between X_1 and X_2 is defined as

$$\rho(X_1, X_2) = \frac{\sigma(X_1, X_2)}{\sigma(X_1)\sigma(X_2)}$$

The Gains From Diversification

The covariance

$$\sigma(X_1, X_2) = E\{[X_1 - E(X_1)][X_2 - E(X_2)]\}$$

is positive if

$$X_1 - E(X_1) \text{ and } X_2 - E(X_2)$$

tend to have the same sign, negative

$$X_1 - E(X_1) \text{ and } X_2 - E(X_2)$$

tend to have opposite signs, and zero if

$$X_1 - E(X_1) \text{ and } X_2 - E(X_2)$$

show no tendency to have the same or opposite signs.

The Gains From Diversification

Mathematically, therefore, the covariance

$$\sigma(X_1, X_2) = E\{[X_1 - E(X_1)][X_2 - E(X_2)]\}$$

measures the extent to which the two random variables tend to move together.

Economically, buying two assets with returns that are imperfectly, and especially, negatively correlated is like buying insurance: one return will be high when the other is low and vice versa, reducing the overall risk of the portfolio.

The Gains From Diversification

The correlation

$$\rho(X_1, X_2) = \frac{\sigma(X_1, X_2)}{\sigma(X_1)\sigma(X_2)}$$

has the same sign as the covariance, and is therefore also a measure of co-movement.

But "scaling" the covariance by the two standard deviations makes the correlation range between -1 and 1:

$$-1 \leq \rho(X_1, X_2) \leq 1$$

The Gains From Diversification

Hence

$$\begin{aligned}\sigma_P^2 &= w^2 E[(\tilde{r}_1 - \mu_1)^2] + (1-w)^2 E[(\tilde{r}_2 - \mu_2)^2] \\ &\quad + 2w(1-w)E[(\tilde{r}_1 - \mu_1)(\tilde{r}_2 - \mu_2)]\end{aligned}$$

implies

$$\begin{aligned}\sigma_P^2 &= w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_{12} \\ &= w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_1\sigma_2\rho_{12}\end{aligned}$$

where

$\sigma_{12} =$ the covariance between $\tilde{r}_1$ and $\tilde{r}_2$

$\rho_{12} =$ the correlation between $\tilde{r}_1$ and $\tilde{r}_2$

The Gains From Diversification

This is the source of the gains from diversification: the expected portfolio return

$$\mu_P = w\mu_1 + (1-w)\mu_2$$

is a weighted average of the expected returns on the individual asset returns, but the standard deviation of the portfolio return

$$\sigma_P = [w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_1\sigma_2\rho_{12}]^{1/2}$$

is not a weighted average of the standard deviations of the returns on the individual assets and can be reduced by choosing a mix of assets ($0 < w < 1$) when ρ_{12} is less than one and, especially, when ρ_{12} is negative.

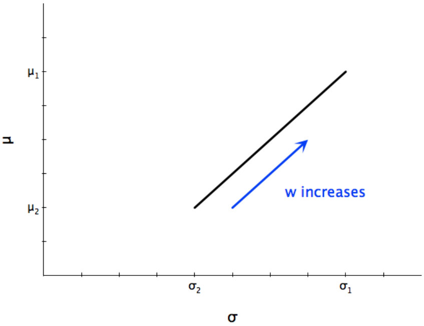
The Gains From Diversification

To see more specifically how this works, start with the case where $\rho_{12} = 1$ so that the individual asset returns are perfectly correlated. This is the one case in which there are no gains from diversification. With $\rho_{12} = 1$,

$$\begin{aligned}
\sigma_P &= [w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_1\sigma_2\rho_{12}]^{1/2} \\
&= [w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_1\sigma_2]^{1/2} \\
&= \{[w\sigma_1 + (1-w)\sigma_2]^2\}^{1/2} \\
&= |w\sigma_1 + (1-w)\sigma_2|.
\end{aligned}$$

In this special case, the standard deviation of the return on the portfolio is a weighted average of the standard deviations of the returns on the individual assets.

The Gains From Diversification



When $\rho_{12} = 1$, so that individual asset returns are perfectly correlated, there are no gains from diversification.

The Gains From Diversification

Next, let's consider the opposite extreme, in which $\rho_{12} = -1$ so that the individual asset returns are perfectly, but negatively, correlated:

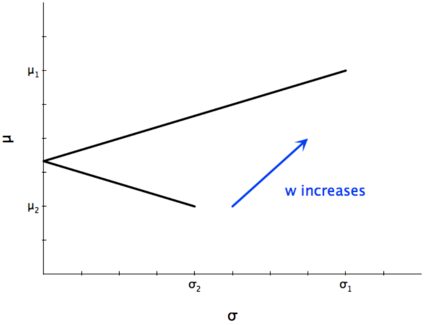
$$
\begin{aligned}
\sigma_P &= [w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_1\sigma_2\rho_{12}]^{1/2} \\
&= [w^2\sigma_1^2 + (1-w)^2\sigma_2^2 - 2w(1-w)\sigma_1\sigma_2]^{1/2} \\
&= \{[w\sigma_1 - (1-w)\sigma_2]^2\}^{1/2} \\
&= |w\sigma_1 - (1-w)\sigma_2|.
\end{aligned}
$$

In this special case, the setting

$$w = \frac{\sigma_2}{\sigma_1 + \sigma_2}$$

creates a "synthetic" risk free portfolio!

The Gains From Diversification



When $\rho_{12} = -1$, so that individual asset returns are perfectly, but negatively correlated, risk can be eliminated via diversification.

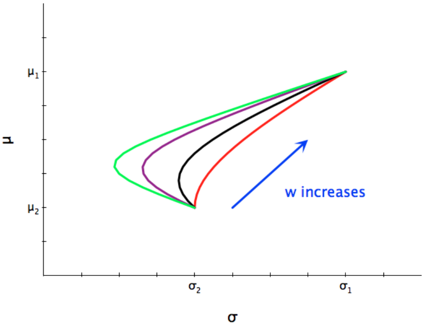
The Gains From Diversification

$$\mu_P = w\mu_1 + (1-w)\mu_2$$

$$\sigma_P = [w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_1\sigma_2\rho_{12}]^{1/2}$$

In all intermediate cases, there will still be gains from diversification. These gains will become stronger as ρ_{12} declines from 1 to -1.

The Gains From Diversification



As ρ_{12} decreases from 0.5 to 0 to -0.5 to -0.75, the gains from diversification strengthen.

The Gains from Diversification

$$\tilde{r}_p = w\tilde{r}_1 + (1-w)\tilde{r}_2$$

$$\mu_P = w\mu_1 + (1-w)\mu_2$$

$$\sigma_P = [w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_1\sigma_2\rho_{12}]^{1/2}$$

PS11: Suppose $\mu_1 = 8$, $\mu_2 = 4$, $\sigma_1 = 8$, and $\sigma_2 = 4$. Calculate μ_p and σ_p for various values of w when $\rho_{12} = 0$ and $\rho_{12} = -0.5$. The gains from diversification are strongest when ρ_{12} is negative, but still present whenever $\rho_{12} < 1$.