ECON 337901
FINANCIAL ECONOMICS

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March 29, 2022

5 Risk Aversion and Investment Decisions

A Risk Aversion and Portfolio Allocation

B Portfolios, Risk Aversion, and Wealth

Risk Aversion and Portfolio Allocation

Let's now put our framework of decision-making under uncertainty to use.

Consider a risk-averse investor with vN-M expected utility who divides his or her initial wealth Y_0 into an amount a allocated to a risky asset – say, the stock market – and an amount $Y_0 - a$ allocated to a safe asset – say, a bank account or a government bond.

Risk Aversion and Portfolio Allocation

Y_0 = initial wealth

a = amount allocated to stocks

$\tilde{r}$ = random return on stocks

r_f = risk-free return

$\tilde{Y}_1$ = terminal wealth

$$\begin{aligned}\tilde{Y}_1 &= (1+r_f)(Y_0-a)+a(1+\tilde{r}) \\ &= Y_0(1+r_f)+a(\tilde{r}-r_f)\end{aligned}$$

Risk Aversion and Portfolio Allocation

The investor chooses a to maximize expected utility:

$$\max_a E[u(\tilde{Y}_1)] = \max_a E\{u[Y_0(1+r_f) + a(\tilde{r} - r_f)]\}$$

The first-order condition is

$$E\{u'[Y_0(1+r_f) + a^*(\tilde{r} - r_f)](\tilde{r} - r_f)\} = 0.$$

Note: we are allowing the investor to sell stocks short ($a^* < 0$) or to buy stocks on margin ($a^* > Y_0$) if he or she desires.

Risk Aversion and Portfolio Allocation

The following results were proven by Kenneth Arrow in "The Theory of Risk Aversion," published in the 1971 volume *Essays in the Theory of Risk-Bearing* and reprinted in 1983 in volume 3 of the *Collected Papers of Kenneth J. Arrow* (Harvard University Press).

Risk Aversion and Portfolio Allocation

Theorem If the Bernoulli utility function u is increasing and concave, then

$$a^* > 0 \text{ if and only if } E(\tilde{r}) > r_f$$

$$a^* = 0 \text{ if and only if } E(\tilde{r}) = r_f$$

$$a^* < 0 \text{ if and only if } E(\tilde{r}) < r_f$$

Thus, a risk-averse investor will always allocate at least some funds to the stock market if the expected return on stocks exceeds the risk-free rate.

Risk Aversion and Portfolio Allocation

Danthine and Donaldson (3rd ed., p.41) report that in the United States, 1889-2010, average real (inflation-adjusted) returns on stocks and risk-free bonds are

$$E(\tilde{r}) = 0.075 \text{ (7.5 percent per year)}$$
$$r_f = 0.011 \text{ (1.1 percent per year)}$$

The equity risk premium of $E(\tilde{r}) - r_f = 0.064$ (6.4 percent) is not only positive, it is huge. The implication of the theory is that all investors, even the most risk averse, should have some money invested in the stock market.

Risk Aversion and Portfolio Allocation

Using updated data (1871-2021) on the S&P 500, adjusted for dividends and inflation from Robert Shiller's webpage:

`http://www.econ.yale.edu/~shiller/data.htm`

$$E(\tilde{r}) = 0.086 \qquad \sigma(\tilde{r}) = 0.178$$

$$E(\tilde{r}) - 2\sigma(\tilde{r}) = -0.270 \qquad E(\tilde{r}) + 2\sigma(\tilde{r}) = 0.442$$

$$r_{1931} = -0.380 \qquad r_{1933} = 0.531$$
$$r_{2008} = -0.356 \qquad r_{2009} = 0.298$$
$$r_{2018} = -0.062 \qquad r_{2019} = 0.250$$
$$r_{2020} = 0.162 \qquad r_{2021} = 0.144$$

Risk Aversion and Portfolio Allocation

Arrow also showed that a^* rises when either $R_A(Y_0)$ or $R_R(Y_0)$ goes down.

If there are investors with the same initial wealth but different coefficients of risk aversion, the investor who is less risk averse will hold more stocks.

Risk Aversion and Portfolio Allocation

Arrow also showed that if an investor has:

1. Decreasing absolute risk aversion, then a^* rises with Y_0.
2. Constant absolute risk aversion, then a^* does not depend on Y_0.
3. Increasing absolute risk aversion, then a^* falls with Y_0.

Intuitively, if $R_A(Y_0)$ is constant (does not depend on Y_0), then the investor finds his or her "optimal absolute bet" a^* and sticks with it, even as income goes up.

Risk Aversion and Portfolio Allocation

Arrow also showed that if an investor has:

1. Decreasing relative risk aversion, then a^*/Y_0 rises with Y_0.
2. Constant relative risk aversion, then a^*/Y_0 does not depend on Y_0.
3. Increasing relative risk aversion, then a^*/Y_0 falls with Y_0.

Intuitively, if $R_R(Y_0)$ is constant (does not depend on Y_0), then the investor finds his or her "optimal bet" a^*/Y_0 as a fraction of income, and lets a^* rise proportionally as his or her income goes up.

Risk Aversion and Portfolio Allocation

As an example, suppose $u(Y) = \ln(Y)$, as suggested by Daniel Bernoulli. Recall that for this utility function, $u'(Y) = 1/Y$. Then assume that stock returns can either be good or bad:

$$\tilde{r} = \begin{cases} r_G & \text{with probability } \pi \\ r_B & \text{with probability } 1-\pi \end{cases}$$

where $r_G > r_f > r_B$ defines the "good" and "bad" states and

$$\pi r_G + (1-\pi) r_B > r_f,$$

so that $E(\tilde{r}) > r_f$ and the investor will choose $a^* > 0$.

Risk Aversion and Portfolio Allocation

The problem

$$\max_a E\{u[Y_0(1+r_f)+a(\tilde{r}-r_f)]\}$$

specializes to

$$\begin{aligned}\max_a \quad & \pi \ln[Y_0(1+r_f)+a(r_G-r_f)] \\ & + (1-\pi)\ln[Y_0(1+r_f)+a(r_B-r_f)]\end{aligned}$$

Risk Aversion and Portfolio Allocation

Problem Set 11, Question 1: Specialize the problem

$$\max_a \quad \pi \ln[Y_0(1+r_f) + a(r_G - r_f)] + (1-\pi)\ln[Y_0(1+r_f) + a(r_B - r_f)]$$

by setting $Y_0 = 100$, $r_f = 0.10$, $r_G = 0.30$, $r_B = 0.05$, and $\pi = 1 - \pi = 1/2$.

The problem becomes

$$\max_a (1/2)\ln(110 + 0.20a) + (1/2)\ln(110 - 0.05a)$$

To find a^*, differentiate with respect to a using the chain rule and set the result equal to zero.

Risk Aversion and Portfolio Allocation

The general problem

$$\max_{a} \quad \pi \ln[Y_0(1+r_f) + a(r_G - r_f)] + (1-\pi)\ln[Y_0(1+r_f) + a(r_B - r_f)]$$

has first-order condition

$$\frac{\pi(r_G - r_f)}{Y_0(1+r_f) + a^*(r_G - r_f)} + \frac{(1-\pi)(r_B - r_f)}{Y_0(1+r_f) + a^*(r_B - r_f)} = 0.$$

Risk Aversion and Portfolio Allocation

$$\frac{\pi(r_G - r_f)}{Y_0(1+r_f) + a^*(r_G - r_f)} + \frac{(1-\pi)(r_B - r_f)}{Y_0(1+r_f) + a^*(r_B - r_f)} = 0$$

$$\begin{aligned} & \pi(r_G - r_f)[Y_0(1+r_f) + a^*(r_B - r_f)] \\ = \; & -(1-\pi)(r_B - r_f)[Y_0(1+r_f) + a^*(r_G - r_f)] \end{aligned}$$

$$\begin{aligned} & a^*(r_G - r_f)(r_B - r_f) \\ = \; & -Y_0(1+r_f)[\pi(r_G - r_f) + (1-\pi)(r_B - r_f)] \end{aligned}$$

Risk Aversion and Portfolio Allocation

$$\begin{aligned} & a^*(r_G - r_f)(r_B - r_f) \\ = \; & -Y_0(1 + r_f)[\pi(r_G - r_f) + (1 - \pi)(r_B - r_f)] \end{aligned}$$

implies

$$\frac{a^*}{Y_0} = -\frac{(1 + r_f)[\pi(r_G - r_f) + (1 - \pi)(r_B - r_f)]}{(r_G - r_f)(r_B - r_f)},$$

which is positive, since $r_G > r_f > r_B$ and

$$E(\tilde{r}) - r_f = \pi(r_G - r_f) + (1 - \pi)(r_B - r_f) > 0.$$

Risk Aversion and Portfolio Allocation

$$\frac{a^*}{Y_0} = -\frac{(1+r_f)[\pi(r_G - r_f) + (1-\pi)(r_B - r_f)]}{(r_G - r_f)(r_B - r_f)},$$

In this case, a^*:

Rises proportionally with Y_0.

Increases as $E(\tilde{r}) - r_f$ rises.

Falls as r_G and r_B move father away from r_f, holding $E(\tilde{r})$ constant; that is, in response to a "mean preserving spread."

Risk Aversion and Portfolio Allocation

$$\frac{a^*}{Y_0} = -\frac{(1+r_f)[\pi(r_G - r_f) + (1-\pi)(r_B - r_f)]}{(r_G - r_f)(r_B - r_f)},$$

r_f	r_G	r_B	π	$E(\tilde{r})$	a^*/Y_0
0.05	0.40	−0.20	0.50	0.10	0.60
0.05	0.30	−0.10	0.50	0.10	1.40
0.05	0.40	−0.20	0.75	0.25	2.40

The fraction of initial wealth allocated to stocks rises when stocks become less risky or pay higher expected returns.