ECON 337901
FINANCIAL ECONOMICS

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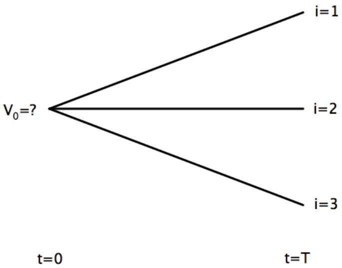
Black-Scholes Option Pricing

Black and Scholes and Merton considered a more general setting, in which the option priced at $t = 0$ does not expire until $t = T$.

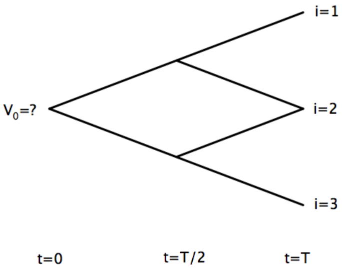
They also allowed for (many) more than two possible states at $t = T$.

Black-Scholes Option Pricing



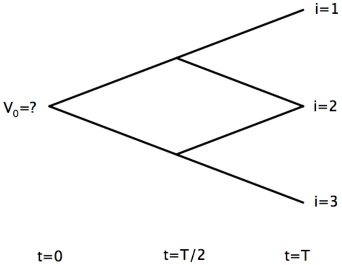
The technical problem is that with more than two states at $t = T$, more than two assets are needed to create a portfolio with the same payoffs as the option.

Black-Scholes Option Pricing



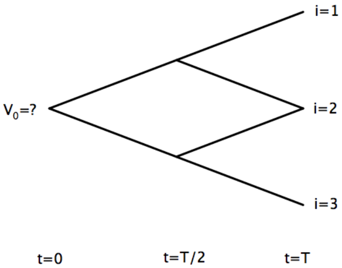
Black and Scholes and Merton realized that this problem can be solved by breaking the full period into sub-periods, so that there are only two states in each sub-period.

Black-Scholes Option Pricing



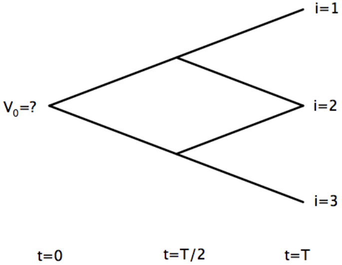
With three states at $t = T$, only two subperiods are needed, but with many states at $t = T$, many subperiods are needed.

Black-Scholes Option Pricing



A dynamic hedging strategy can then be used to track the payoffs on the option using a portfolio consisting only of the stock and bond . . .

Black-Scholes Option Pricing



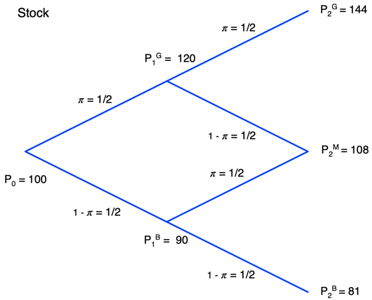
. . . but where the number of shares and the number of bonds must be adjusted in each subperiod so that the portfolio can continue to track the option's payoffs.

Black-Scholes Option Pricing

As an example of how to implement dynamic hedging and price an option with more than two final states, set $T = 2$ and use two subperiods.

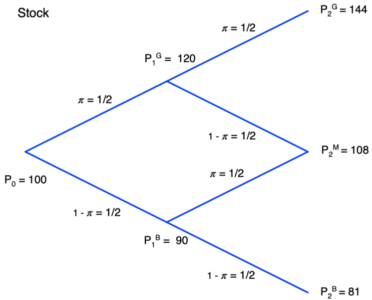
Then there will be three periods $t = 0$, $t = 1$, and $t = 2$, a good and bad state at $t = 1$, and a good, medium, and bad state at $t = 2$.

Black-Scholes Option Pricing



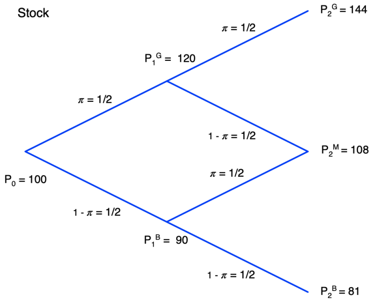
Suppose the stock price starts at $P_0 = 100$, and moves up by 20 percent or down by 10 percent with equal probability in each subperiod.

Black-Scholes Option Pricing



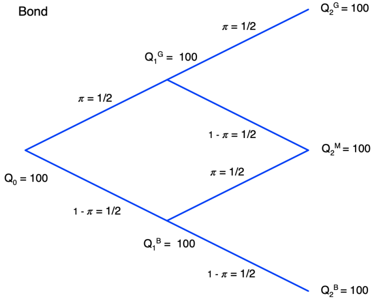
Notice that these assumptions make the middle state more likely than the good or bad at $t = 2$.

Black-Scholes Option Pricing



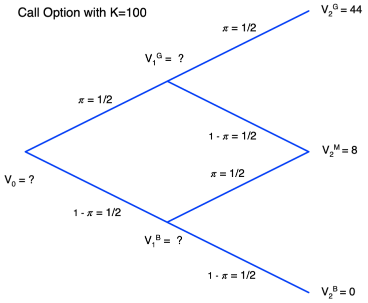
In fact, as the number of subperiods on the binomial tree grows larger, the distribution of final states will start to look more and more like the normal distribution.

Black-Scholes Option Pricing



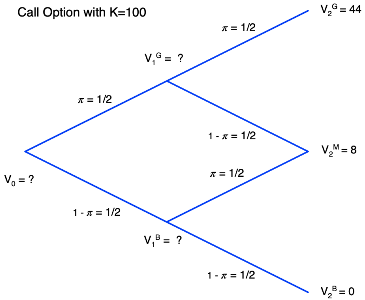
Assume for simplicity that the bond price stays constant at 100, that is, the interest rate is zero.

Black-Scholes Option Pricing



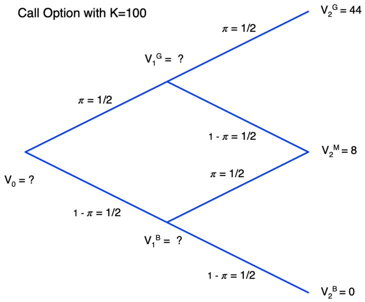
A call option with $K = 100$ and expiration $t = 2$ will be in the money in the good and medium states, but out of the money in the bad state.

Black-Scholes Option Pricing



We can use dynamic hedging and "backwards recursion" to determine the option values V_1^G and V_2^B at $t = 1$ and then V_0 at $t = 0$.

Black-Scholes Option Pricing



Focus first on the good state at $t = 1$.

Black-Scholes Option Pricing

Focus first on the good state at $t = 1$:

The stock price is $P_1^G = 120$ and can rise to $P_2^G = 144$ or fall to $P_2^M = 108$.

The bond price is $Q_1^G = 100$ and remains at $Q_2^G = Q_2^M = 100$ no matter what.

The option price is $V_1^G = ?$ and can rise to $V_2^G = 44$ or fall to $V_2^M = 8$.

Black-Scholes Option Pricing

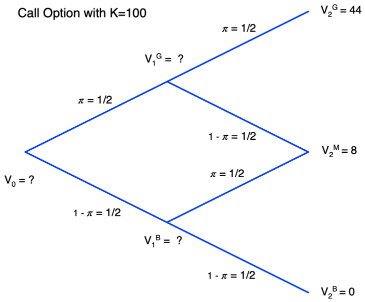
Form a portfolio of s shares and b bonds to replicate the option's payoffs going from the good state at $t = 1$ to either the good or medium state at $t = 2$:

$$44 = 144s + 100b$$
$$8 = 108s + 100b$$

No arbitrage then requires

$$V_1^G = P_1^G s + Q_1^G s = 120s + 100b.$$

Black-Scholes Option Pricing



Now move down to the bad state at $t = 1$.

Black-Scholes Option Pricing

Move down to the bad state at $t = 1$:

The stock price is $P_1^B = 90$ and can rise to $P_2^M = 108$ or fall to $P_2^B = 81$.

The bond price is $Q_1^B = 100$ and remains at $Q_2^M = Q_2^B = 100$ no matter what.

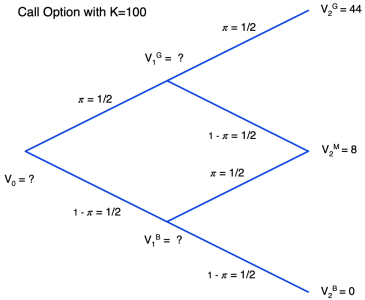
The option price is $V_1^B = ?$ and can rise to $V_2^M = 8$ or fall to $V_2^B = 0$.

Black-Scholes Option Pricing

Form a portfolio of s shares and b bonds to replicate the option's payoffs going from the bad state at $t = 1$ to either the medium or bad state at $t = 2$.

Then find the option price V_1^B in the bad state at $t = 1$ implied by no arbitrage.

Black-Scholes Option Pricing



Finally, move back to $t = 0$, having filled in the values for V_1^G and V_1^B.

Black-Scholes Option Pricing

Move back to $t = 0$:

The stock price is $P_0 = 100$ and can rise to $P_1^G = 120$ or fall to $P_1^B = 90$.

The bond price is $Q_0 = 100$ and remains at $Q_1^G = Q_1^B = 100$ no matter what.

The option price is $V_0 = ?$ and can rise to V_1^G or fall to V_1^B.

Black-Scholes Option Pricing

Form a portfolio of s shares and b bonds to replicate the option's payoffs going from $t = 0$ to either the good or bad state at $t = 1$.

Then find the option price V_0 at $t = 0$ implied by no arbitrage.

Black-Scholes Option Pricing

Black and Scholes used methods in stochastic calculus developed by Kiyoshi Ito (Japan, 1915-2008) in the 1940s and early 1950s to "take the limit" as the number of subperiods and the number of final states become infinite and the distribution of final states becomes normal.

The derivation is tedious and the formula looks daunting at first glance. But we can get a feel for the intuition with reference to dynamic hedging and the binomial tree.

Black-Scholes Option Pricing

Let T denote the number of years until the option expires (can be a fraction) and σ denote the standard deviation of the stock's annual return (must be estimated).

Then

$$q^o = N_1 q^s - N_2 q^b K = N_1 q^s - N_2 \left(\frac{K}{1 + r_f} \right)$$

showing that the option price q^o equals the cost of assembling a portfolio consisting of a long position $s = N_1$ shares and a short position of $b = -N_2 K$ bonds.

Black-Scholes Option Pricing

In particular

$$q^o = N_1 q^s - N_2 q^b K = N_1 q^s - N_2 \left(\frac{K}{1 + r_f} \right)$$

where $N_1 = F(d_1)$ and $N_2 = F(d_2)$,

$$d_1 = \frac{\ln(q^s/K) + (r_f + \sigma^2/2)T}{\sigma\sqrt{T}} \text{ and } d_2 = d_1 - \sigma\sqrt{T}$$

and F is the standard normal cumulative distribution function, so that $F(X)$ measures the probability that a random variable that is normally distributed with mean zero and variance one turns out to be less than or equal to X.

Black-Scholes Option Pricing

To use the Black-Scholes formula

$$q^0 = N_1 q^s - N_2 q^b K = N_1 q^s - N_2 \left(\frac{K}{1 + r_f} \right)$$

where $N_1 = F(d_1)$ and $N_2 = F(d_2)$,

$$d_1 = \frac{\ln(q^s/K) + (r_f + \sigma^2/2)T}{\sigma\sqrt{T}} \text{ and } d_2 = d_1 - \sigma\sqrt{T}$$

you need an estimate of σ, the standard deviation of the return on the stock.

Black-Scholes Option Pricing

Alternatively, if you see the price of a traded option, you can use the Black-Scholes formula

$$q^0 = N_1 q^s - N_2 q^b K = N_1 q^s - N_2 \left(\frac{K}{1 + r_f} \right)$$

where $N_1 = F(d_1)$ and $N_2 = F(d_2)$,

$$d_1 = \frac{\ln(q^s/K) + (r_f + \sigma^2/2)T}{\sigma\sqrt{T}} \text{ and } d_2 = d_1 - \sigma\sqrt{T}$$

to estimate σ, the standard deviation of the return on the stock. In fact, the VIX volatility index is based on the σ implied by the prices of options on the S&P 500.

Using Options to Infer Contingent Claims Prices

Notice that call options often have payoff structures that resemble those of contingent claims: making positive payoffs in "good" states and expiring worthless in "bad" states.

Douglas Breeden (US, b.1950) and Robert Litzenberger (US, b.1943) devised a way of using option prices to infer contingent claims prices, allowing for many states of the world.

Douglas Breeden and Robert Litzenberger, "Prices of State-Contingent Claims Implicit in Option Prices," *Journal of Business* Vol.51 (October 1978): pp.621-651.

Using Options to Infer Contingent Claims Prices

Suppose there are N states, corresponding to different levels of the S&P 500, with

$$P^1 < P^2 < \ldots < P^N$$

and

$$P^{i+1} = P^i + \delta$$

with $\delta > 0$.

Using Options to Infer Contingent Claims Prices

For each state i, construct a "butterfly" portfolio of call options:

Buy one calls with strike price P^{i-1}

Write (sell short) two calls with strike price P^{i}

Buy one call with strike price P^{i+1}

If q_o^i is the price of an option with strike price P^i, this portfolio costs

$$q_o^{i-1} + q_o^{i+1} - 2q_o^i$$

Using Options to Infer Contingent Claims Prices

Now let's compute the portfolio's payoffs:

S&P 500	Long $K = P^{i-1}$	Short 2 $K = P^{i}$	Long $K = P^{i+1}$	Total
$P \leq P^{i-1}$	0	0	0	0
$P = P^{i}$	$P^{i} - P^{i-1}$	0	0	δ
$P \geq P^{i+1}$	$P - P^{i-1}$	$-2(P - P^{i})$	$P - P^{i+1}$	0

$$
\begin{aligned}
& (P - P^{i-1}) - 2(P - P^{i}) + (P - P^{i+1}) \\
= \;& 2P^{i} - P^{i-1} - P^{i+1} \\
= \;& 2P^{i} - (P^{i} - \delta) - (P^{i} + \delta) = 0
\end{aligned}
$$

Using Options to Infer Contingent Claims Prices

S&P 500	Long $K = P^{i-1}$	Short 2 $K = P^{i}$	Long $K = P^{i+1}$	Total
$P \leq P^{i-1}$	0	0	0	0
$P = P^{i}$	$P^{i} - P^{i-1}$	0	0	δ
$P \geq P^{i+1}$	$P - P^{i-1}$	$-2(P - P^{i})$	$P - P^{i+1}$	0

Since the portfolio's payoffs replicate those from δ claims for state i, the price q^{i}_{cc} of a contingent claim for state i must satisfy

$$q^{i}_{cc} = (1/\delta)(q^{i-1}_{o} + q^{i+1}_{o} - 2q^{i}_{o})$$

Using Options to Infer Contingent Claims Prices

$$q^i_{cc} = (1/\delta)(q^{i-1}_o + q^{i+1}_o - 2q^i_o)$$

Additional accuracy can be achieved by choosing smaller values of δ, that is, by using a "finer grid" to define the states.

Options trade on the S&P 500 with many strike prices, so data on q^i_o are readily available.

Note that you don't have to actually trade the options to price the contingent claims.