ECON 337901
FINANCIAL ECONOMICS

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March 1, 2022

Black-Scholes Option Pricing

A call option is a contract that gives the buyer the right, but not the obligation, to purchase a share of stock at the strike price K at $t = 1$.

At $t = 1$, the call is said to be in the money if the actual share price is above the strike price and out of the money if the actual share price is below the strike price.

At $t = 1$, the option will have value only if it is in the money. But at $t = 0$, the option will have value even if there is only a probability of it being in the money at $t = 1$.

Black-Scholes Option Pricing

Fischer Black (US, 1938-1995) and Myron Scholes (Canada/US, b.1941, Nobel Prize 1997) were the first to derive a formula for the price of an option.

Robert Merton (US, b.1944, Nobel Prize 1997) arrived at the same formula in a simpler way, by showing how options prices could be inferred from assumptions about and observations on the underlying stock price.

Black-Scholes Option Pricing

The arguments used by Merton were not exactly those from Arrow-Debreu no-arbitrage theory that would use the price of the stock and bond to infer contingent claims prices, then use contingent claims prices to compute the price of the option.

But his analysis followed along similar lines, and today it is recognized that one could use the Arrow-Debreu approach to obtain the same results.

Black-Scholes Option Pricing

Their papers were both published in 1973.

Fischer Black and Myron Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* Vol.81 (May-June 1973): pp.637-654.

Robert Merton, "Theory of Rational Option Pricing," *The Bell Journal of Economics and Management Science* Vol.4 (Spring 1973): pp.141-183.

Black-Scholes Option Pricing

To see how the theory works, assume a simple two-period structure, with $t = 0$ and $t = 1$, and assume as well, that there are only two states, $i = G$ and $i = B$, at $t = 1$. Let

$$q^s = \text{ price of the stock at } t = 0$$

$$P^G = \text{ price of the stock in state } i = G \text{ at } t = 1$$

$$P^B = \text{ price of the stock in state } i = B \text{ at } t = 1$$

Black-Scholes Option Pricing

Likewise, let

$$q^b = \text{ price of the bond at } t = 0$$

$$1 = \text{ payoff from bond at } i = G \text{ at } t = 1$$

$$1 = \text{ payoff from bond at } i = B \text{ at } t = 1$$

Black-Scholes Option Pricing

Now consider a call option on the stock with strike price K. Let

$$q^o = \text{ price of the call at } t = 0$$

$$C^G = \text{ payoff generated by the call in state } i = G \text{ at } t = 1$$

$$C^B = \text{ payoff generated by the call in state } i = B \text{ at } t = 1$$

Assume, for now, that the call is in the money in both states at $t = 1$. Then:

$$C^G = P^G - K \text{ and } C^B = P^B - K$$

Black-Scholes Option Pricing

One of the key insights that underlies the Black-Scholes formula is that we don't need to make any specific assumptions about risk or risk aversion to price the option.

Instead, we can use a no-arbitrage argument that:

1. Replicates the option's payoffs using a portfolio of the stock and the risk-free bond.
2. Values the option based on the cost of assembling the portfolio.

Black-Scholes Option Pricing

State	Stock's Payoff	Bond's Payoff	Option's Payoff
G	P^G	1	$P^G - K$
B	P^B	1	$P^B - K$

We want to construct a portfolio consisting of s shares of the stock and b bonds that replicates the payoffs from the option in both states at $t = 1$:

$$sP^G + b = P^G - K$$

$$sP^B + b = P^B - K$$

Black-Scholes Option Pricing

$$sP^G + b = P^G - K$$
$$sP^B + b = P^B - K$$

This is a set of two linear equations in the two unknowns: s and b. The solution is

$$s = 1 \text{ and } b = -K$$

Since the stock costs q^s and the bond costs q^b, the cost of this portfolio at $t = 0$ is

$$q^s - q^b K$$

Black-Scholes Option Pricing

The option's payoffs are replicated by a portfolio with

$$s = 1 \text{ and } b = -K$$

and since the stock costs q^s and the bond costs q^b, the cost of this portfolio at $t = 0$ is

$$q^s - q^b K$$

But this means that the price of the option must also be

$$q^o = q^s - q^b K$$

Black-Scholes Option Pricing

Next, let's consider the case in which the call is in the money in the good state and out of the money in the bad state at $t = 1$.

Then

$$C^G = P^G - K \text{ and } C^B = 0$$

Black-Scholes Option Pricing

State	Stock's Payoff	Bond's Payoff	Option's Payoff
G	P^G	1	$P^G - K$
B	P^B	1	0

Again we want to construct a portfolio consisting of s shares of the stock and b bonds that replicates the payoffs from the option in both states at $t = 1$:

$$sP^G + b = P^G - K$$

$$sP^B + b = 0$$

Black-Scholes Option Pricing

$$sP^G + b = P^G - K$$

$$sP^B + b = 0$$

Again this is a set of two linear equations in the two unknowns: s and b. The solution is

$$s = \frac{P^G - K}{P^G - P^B} \text{ and } b = -\frac{P^B(P^G - K)}{P^G - P^B}$$

Since the stock costs q^s and the bond costs q^b, the cost of this portfolio at $t = 0$ is

$$\left(\frac{P^G - K}{P^G - P^B}\right) q^s + \left[-\frac{P^B(P^G - K)}{P^G - P^B}\right] q^b$$

Black-Scholes Option Pricing

But since the portfolio of the stock and bond again replicates the payoffs from the option, this implies that the option's price must be

$$\begin{aligned} q^o &= \left(\frac{P^G - K}{P^G - P^B}\right) q^s + \left[-\frac{P^B(P^G - K)}{P^G - P^B}\right] q^b \\ &= \frac{(q^s - q^b P^B)(P^G - K)}{P^G - P^B} \end{aligned}$$

Black-Scholes Option Pricing

Finally, there is the easy case in which the call is out of the money in both states at $t = 1$.

Then

$$C^G = 0 \text{ and } C^B = 0$$

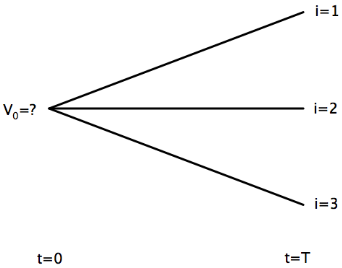
The option's payoffs can be replicated by a portfolio consisting of zero shares of the stock and zero bonds, which costs zero at $t = 0$. Equivalently, an asset that pays off nothing should cost nothing.

Black-Scholes Option Pricing

Black and Scholes and Merton considered a more general setting, in which the option priced at $t = 0$ does not expire until $t = T$.

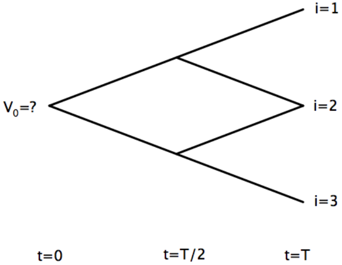
They also allowed for (many) more than two possible states at $t = T$.

Black-Scholes Option Pricing



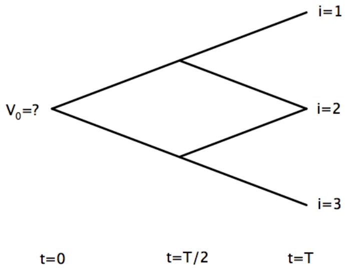
The technical problem is that with more than two states at $t = T$, more than two assets are needed to create a portfolio with the same payoffs as the option.

Black-Scholes Option Pricing



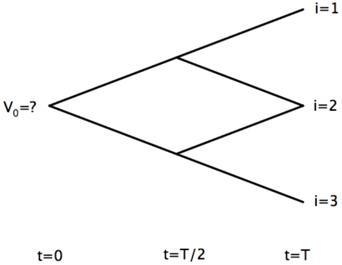
Black and Scholes and Merton realized that this problem can be solved by breaking the full period into sub-periods, so that there are only two states in each sub-period.

Black-Scholes Option Pricing



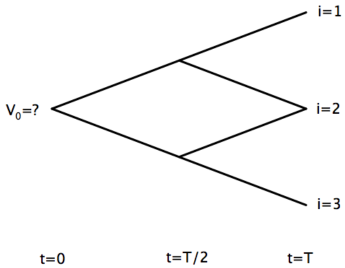
With three states at $t = T$, only two subperiods are needed, but with many states at $t = T$, many subperiods are needed.

Black-Scholes Option Pricing



A dynamic hedging strategy can then be used to track the payoffs on the option using a portfolio consisting only of the stock and bond . . .

Black-Scholes Option Pricing



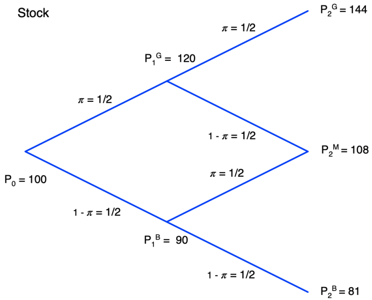
... but where the number of shares and the number of bonds must be adjusted in each subperiod so that the portfolio can continue to track the option's payoffs.

Black-Scholes Option Pricing

As an example of how to implement dynamic hedging and price an option with more than two final states, set $T = 2$ and use two subperiods.

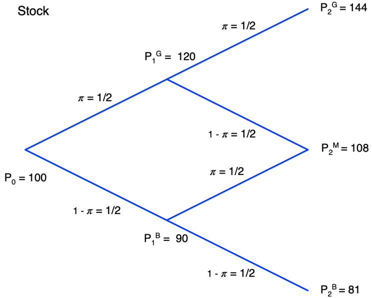
Then there will be three periods $t = 0$, $t = 1$, and $t = 2$, a good and bad state at $t = 1$, and a good, medium, and bad state at $t = 2$.

Black-Scholes Option Pricing



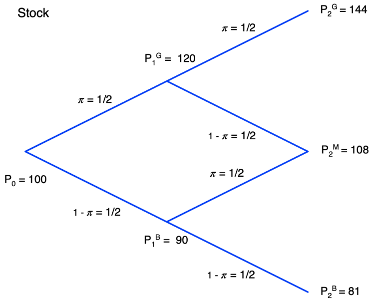
Suppose the stock price starts at $P_0 = 100$, and moves up by 20 percent or down by 10 percent with equal probability in each subperiod.

Black-Scholes Option Pricing



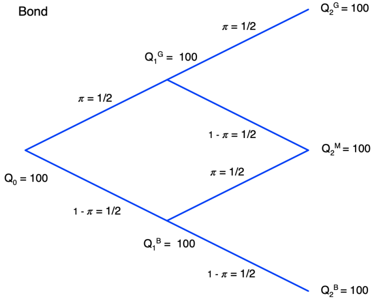
Notice that these assumptions make the middle state more likely than the good or bad at $t = 2$.

Black-Scholes Option Pricing



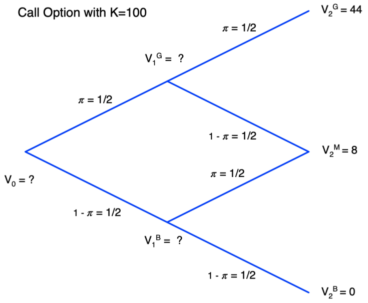
In fact, as the number of subperiods on the binomial tree grows larger, the distribution of final states will start to look more and more like the normal distribution.

Black-Scholes Option Pricing



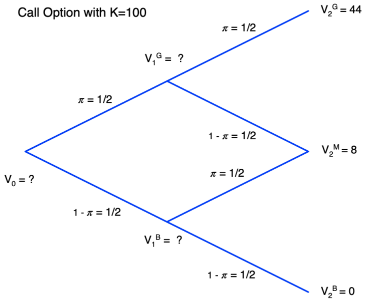
Assume for simplicity that the bond price stays constant at 100, that is, the interest rate is zero.

Black-Scholes Option Pricing



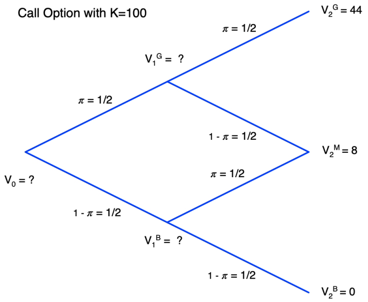
A call option with $K = 100$ and expiration $t = 2$ will be in the money in the good and medium states, but out of the money in the bad state.

Black-Scholes Option Pricing



We can use dynamic hedging and “backwards recursion” to determine the option values V_1^G and V_2^B at $t = 1$ and then V_0 at $t = 0$.

Black-Scholes Option Pricing



Focus first on the good state at $t = 1$.

Black-Scholes Option Pricing

Focus first on the good state at $t = 1$:

The stock price is $P_1^G = 120$ and can rise to $P_2^G = 144$ or fall to $P_2^M = 108$.

The bond price is $Q_1^G = 100$ and remains at $Q_2^G = Q_2^M = 100$ no matter what.

The option price is $V_1^G = ?$ and can rise to $V_2^G = 44$ or fall to $V_2^M = 8$.

Black-Scholes Option Pricing

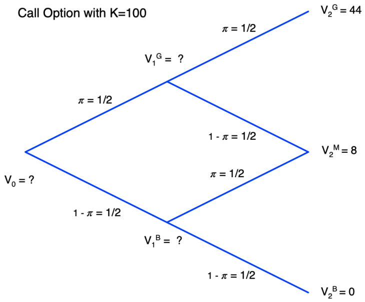
Form a portfolio of s shares and b bonds to replicate the option's payoffs going from the good state at $t = 1$ to either the good or medium state at $t = 2$:

$$44 = 144s + 100b$$
$$8 = 108s + 100b$$

No arbitrage then requires

$$V_1^G = P_1^G s + Q_1^G s = 120s + 100b.$$

Black-Scholes Option Pricing



Now move down to the bad state at $t = 1$.

Black-Scholes Option Pricing

Move down to the bad state at $t = 1$:

The stock price is $P_1^B = 90$ and can rise to $P_2^M = 108$ or fall to $P_2^B = 81$.

The bond price is $Q_1^B = 100$ and remains at $Q_2^M = Q_2^B = 100$ no matter what.

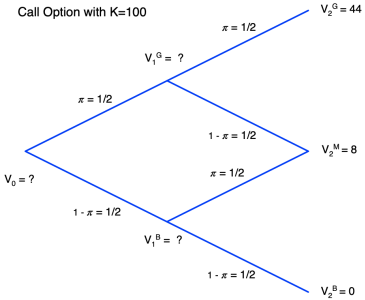
The option price is $V_1^B = ?$ and can rise to $V_2^M = 8$ or fall to $V_2^B = 0$.

Black-Scholes Option Pricing

Form a portfolio of s shares and b bonds to replicate the option's payoffs going from the bad state at $t = 1$ to either the medium or bad state at $t = 2$.

Then find the option price V_1^B in the bad state at $t = 1$ implied by no arbitrage.

Black-Scholes Option Pricing



Finally, move back to $t = 0$, having filled in the values for V_1^G and V_1^B.

Black-Scholes Option Pricing

Move back to $t = 0$:

The stock price is $P_0 = 100$ and can rise to $P_1^G = 120$ or fall to $P_1^B = 90$.

The bond price is $Q_0 = 100$ and remains at $Q_1^G = Q_1^B = 100$ no matter what.

The option price is $V_0 = ?$ and can rise to V_1^G or fall to V_1^B.

Black-Scholes Option Pricing

Form a portfolio of s shares and b bonds to replicate the option's payoffs going from $t = 0$ to either the good or bad state at $t = 1$.

Then find the option price V_0 at $t = 0$ implied by no arbitrage.