ECON 337901
FINANCIAL ECONOMICS

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Consumer Optimization: The Risk Dimension

The problem is to choose c_0, c_1^G, and c_1^B to maximize expected utility

$$u(c_0) + \beta \pi u(c_1^G) + \beta(1 - \pi)u(c_1^B),$$

subject to the budget constraint

$$Y_0 + q^G Y_1^G + q^B Y_1^B \geq c_0 + q^G c_1^G + q^B c_1^B.$$

This was Arrow and Debreu's key insight: that finance is like grocery shopping. Mathematically, making decisions over time and under uncertainty is no different from choosing apples, bananas, and pears!

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The Lagrangian is

$$
\begin{aligned}
L \;=\; & u(c_0) + \beta\pi u(c_1^G) + \beta(1-\pi)u(c_1^B) \\
& + \lambda\left(Y_0 + q^G Y_1^G + q^B Y_1^B - c_0 - q^G c_1^G - q^B c_1^B\right),
\end{aligned}
$$

and the first-order conditions are

$$u'(c_0^*) - \lambda^* = 0$$

$$\beta\pi u'(c_1^{G*}) - \lambda^* q^G = 0$$

$$\beta(1-\pi)u'(c_1^{B*}) - \lambda^* q^B = 0$$

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The first-order conditions

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imply that marginal rates of substitution equal relative prices:

$$\frac{u'(c_0^*)}{\beta\pi u'(c_1^{G*})} = \frac{1}{q^G} \text{ and } \frac{u'(c_0^*)}{\beta(1-\pi)u'(c_1^{B*})} = \frac{1}{q^B}$$

$$\text{and } \frac{\pi u'(c_1^{G*})}{(1-\pi)u'(c_1^{B*})} = \frac{q^G}{q^B}.$$

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Do we really observe consumers trading in contingent claims?

Yes, if we think of financial assets as "bundles" of contingent claims.

This insight is also Arrow and Debreu's.

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A “stock” is a risky asset that pays dividend d^G next year in the good state and d^B next year in the bad state.

These payoffs can be replicated by buying d^G contingent claims for the good state and d^B contingent claims for the bad state.

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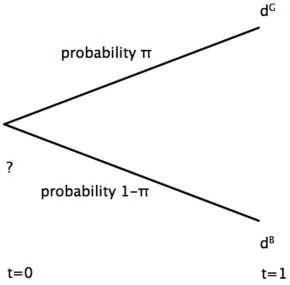

Payoffs for the stock.

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If we start with knowledge of the contingent claims prices q^G and q^B, then we can infer that the stock must sell today for

$$q^{stock} = q^G d^G + q^B d^B.$$

Since if the stock cost more than the equivalent bundle of contingent claims, traders could make profits for sure by short selling the stock and buying the contingent claims; and if the stock cost less than the equivalent bundle of contingent claims, traders could make profits for sure by buying the stock and selling the contingent claims.

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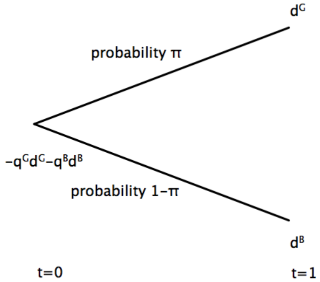

"Pricing" the stock.