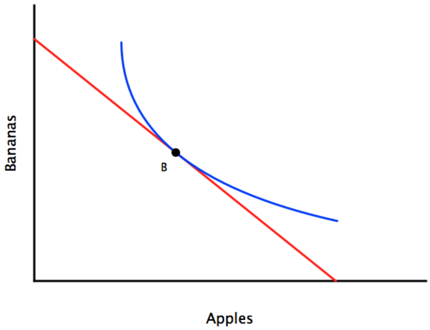
ECON 337901
FINANCIAL ECONOMICS

Peter Ireland

Boston College

February 3, 2022

Consumer Optimization: Graphical Analysis



At B, the optimal choice, the indifference curve is tangent to the budget constraint.

Consumer Optimization: Graphical Analysis

Recall that the budget constraint

$$Y = p_a c_a + p_b c_b$$

or

$$c_b = \frac{Y}{p_b} - \left(\frac{p_a}{p_b}\right) c_a$$

has slope $-(p_a/p_b)$. But what is the slope of the indifference curve?

Consumer Optimization: Graphical Analysis

Suppose that the consumer's preferences are also described by the utility function

$$u(c_a) + \beta u(c_b).$$

The function u is increasing, with $u'(c) > 0$, so that more is preferred to less, and concave, with $u''(c) < 0$, so that marginal utility falls as consumption rises.

The parameter β measures how much more (if $\beta > 1$) or less (if $\beta < 1$) the consumer likes bananas compared to apples.

Consumer Optimization: Graphical Analysis

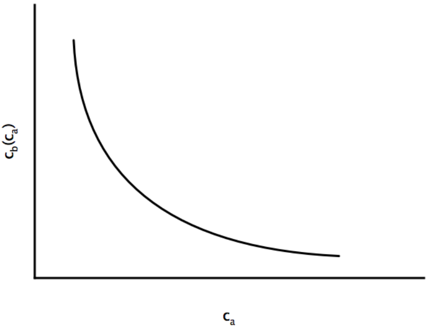
Since an indifference curve traces out the set of (c_a, c_b) combinations that yield a given level of utility $\bar{U}$, the equation for an indifference curve is

$$\bar{U} = u(c_a) + \beta u(c_b).$$

Use this equation to define a new function, $c_b(c_a)$, describing the number of bananas needed, for each number of apples, to keep the consumer on this indifference curve:

$$\bar{U} = u(c_a) + \beta u[c_b(c_a)].$$

Consumer Optimization: Graphical Analysis



The function $c_b(c_a)$ satisfies $\bar{U} = u(c_a) + \beta u[c_b(c_a)]$.

Consumer Optimization: Graphical Analysis

Differentiate both sides of

$$\bar{U} = u(c_a) + \beta u[c_b(c_a)]$$

to obtain

$$0 = u'(c_a) + \beta u'[c_b(c_a)]c_b'(c_a)$$

or

$$c_b'(c_a) = -\frac{u'(c_a)}{\beta u'[c_b(c_a)]}.$$

Consumer Optimization: Graphical Analysis

This last equation,

$$c_b'(c_a) = -\frac{u'(c_a)}{\beta u'[c_b(c_a)]},$$

written more simply as

$$c_b'(c_a) = -\frac{u'(c_a)}{\beta u'(c_b)},$$

measures the slope of the indifference curve: the consumer's marginal rate of substitution.

Consumer Optimization: Graphical Analysis

Thus, the tangency of the budget constraint and indifference curve can be expressed mathematically as

$$\frac{p_a}{p_b} = \frac{u'(c_a)}{\beta u'(c_b)}.$$

The marginal rate of substitution equals the relative prices.

Consumer Optimization: Graphical Analysis

Returning to the more general expression

$$c_b'(c_a) = -\frac{u'(c_a)}{\beta u'[c_b(c_a)]},$$

we can see that $c_b'(c_a) < 0$, so that the indifference curve is downward-sloping, so long as the utility function u is strictly increasing, that is, if more is preferred to less.

Consumer Optimization: Graphical Analysis

$$c_b'(c_a) = -\frac{u'(c_a)}{\beta u'[c_b(c_a)]}$$

Differentiating again yields

$$c_b''(c_a) = -\frac{\beta u'[c_b(c_a)]u''(c_a) - u'(c_a)\beta u''[c_b(c_a)]c_b'(c_a)}{\{\beta u'[c_b(c_a)]\}^2},$$

which is positive if u is strictly increasing (more is preferred to less) and concave (diminishing marginal utility). In this case, the indifference curve will be convex. Again, we see how concave functions have mathematical properties and economic implications that we like.

Consumer Optimization: Algebraic Analysis

Graphical analysis works fine with two goods.

But what about three goods? That depends on how good an artist you are!

And what about four or more goods? Our universe won't accommodate a graph like that!

But once again, calculus makes it easier!

Consumer Optimization: Algebraic Analysis

Consider a consumer who likes three goods:

$$Y = \text{income}$$
$$c_i = \text{consumption of goods } i = 0, 1, 2$$
$$p_i = \text{price of goods } i = 0, 1, 2$$

Suppose the consumer's utility function is

$$u(c_0) + \alpha u(c_1) + \beta u(c_2),$$

where α and β are weights on goods 1 and 2 relative to good 0.

Consumer Optimization: Algebraic Analysis

The consumer chooses c_0, c_1, and c_2 to maximize the utility function

$$u(c_0) + \alpha u(c_1) + \beta u(c_2),$$

subject to the budget constraint

$$Y \geq p_0 c_0 + p_1 c_1 + p_2 c_2.$$

The Lagrangian for this problem is

$$L = u(c_0) + \alpha u(c_1) + \beta u(c_2) + \lambda(Y - p_0 c_0 - p_1 c_1 - p_2 c_2).$$

Consumer Optimization: Algebraic Analysis

$$L = u(c_0) + \alpha u(c_1) + \beta u(c_2) + \lambda(Y - p_0 c_0 - p_1 c_1 - p_2 c_2).$$

First-order conditions:

$$u'(c_0^*) - \lambda^* p_0 = 0$$
$$\alpha u'(c_1^*) - \lambda^* p_1 = 0$$
$$\beta u'(c_2^*) - \lambda^* p_2 = 0$$

Consumer Optimization: Algebraic Analysis

The first-order conditions

$$u'(c_0^*) - \lambda^* p_0 = 0$$

$$\alpha u'(c_1^*) - \lambda^* p_1 = 0$$

$$\beta u'(c_2^*) - \lambda^* p_2 = 0$$

imply

$$\frac{u'(c_0^*)}{\alpha u'(c_1^*)} = \frac{p_0}{p_1} \text{ and } \frac{u'(c_0^*)}{\beta u'(c_2^*)} = \frac{p_0}{p_2} \text{ and } \frac{\alpha u'(c_1^*)}{\beta u'(c_2^*)} = \frac{p_1}{p_2}.$$

The marginal rate of substitution equals the relative prices.

Consumer Optimization: The Time Dimension

Irving Fisher (US, 1867-1947) was the first to recognize that the basic theory of consumer decision-making could be used to understand how to optimally allocate spending intertemporally, that is, over time, as well as how to optimally allocate spending across different goods in a static, or point-in-time, analysis.

Consumer Optimization: The Time Dimension

Following Fisher, return to the case of two goods, but reinterpret:

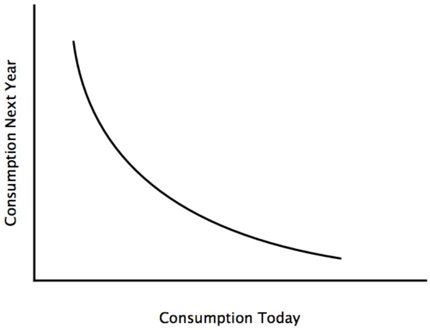
$$c_0 = \text{consumption today}$$
$$c_1 = \text{consumption next year}$$

Suppose that the consumer's utility function is

$$u(c_0) + \beta u(c_1),$$

where β now has a more specific interpretation, as the discount factor, a measure of patience.

Consumer Optimization: The Time Dimension



A concave utility function implies that indifference curves are convex, so that the consumer has a preference for a smoothness in consumption.