

# ECON 337901

# FINANCIAL ECONOMICS

Peter Ireland

Boston College

January 20, 2022

These lecture notes by Peter Ireland are licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0) License. <http://creativecommons.org/licenses/by-nc-sa/4.0/>.

# Course Administration

Course webpage:

<http://irelandp.com/econ3379.html>

Notes/slides, handouts, problems sets and solutions, old exams.

# Course Administration

## Book:

Jean-Pierre Danthine and John Donaldson, *Intermediate Financial Theory*, Second (2005) or Third (2014) edition.

Recommended, but not required.

Third edition is freely available in electronic form via BC libraries.

# Course Administration

## Grading:

Homeworks: 1/3

Take-Home Midterm: 1/3

Take-Home Final: 1/3

# Course Administration

## Academic integrity:

<https://www.bc.edu/content/bc-web/academics/sites/university-catalog/policies-procedures.html>

It's fine to work with other students on the homeworks, but please hand in our own answers for the purposes of grading.

Your work on the midterm and final should be yours and yours alone.

# Mathematical Preliminaries

## Unconstrained Optimization

$$\max_x F(x)$$

## Constrained Optimization

$$\max_x F(x) \text{ subject to } c \geq G(x)$$

# Unconstrained Optimization

To find the value of  $x$  that solves

$$\max_x F(x)$$

you can:

1. Try out every possible value of  $x$ .
2. Use calculus.

Since search could take forever, let's use calculus instead.

# Unconstrained Optimization

Theorem If  $x^*$  solves

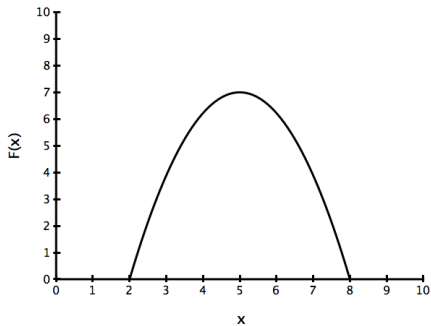
$$\max_x F(x),$$

then  $x^*$  is a **critical point** of  $F$ , that is,

$$F'(x^*) = 0.$$

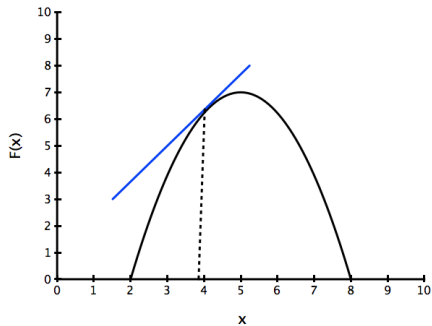


# Unconstrained Optimization



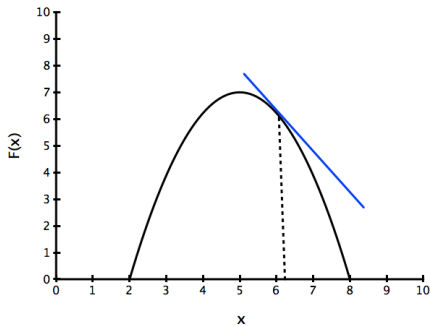
$F(x)$  maximized at  $x^* = 5$

# Unconstrained Optimization



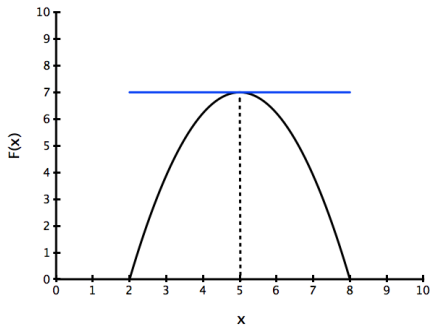
$F'(x) > 0$  when  $x < 5$ .  $F(x)$  can be increased by increasing  $x$ .

# Unconstrained Optimization



$F'(x) < 0$  when  $x > 5$ .  $F(x)$  can be increased by decreasing  $x$ .

# Unconstrained Optimization



$F'(x) = 0$  when  $x = 5$ .  $F(x)$  is maximized.

# Unconstrained Optimization

**Theorem** If  $x^*$  solves

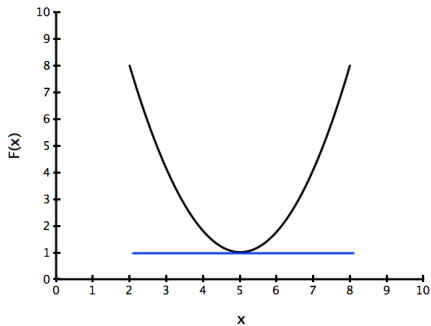
$$\max_x F(x),$$

then  $x^*$  is a **critical point** of  $F$ , that is,

$$F'(x^*) = 0.$$

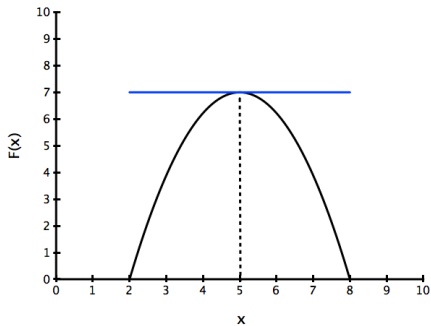
Note that the same **first-order necessary condition**  $F'(x^*) = 0$  also characterizes a value of  $x^*$  that **minimizes**  $F(x)$ .

# Unconstrained Optimization



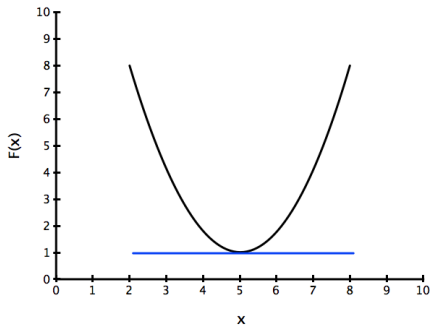
$F'(x) = 0$  when  $x = 5$ .  $F(x)$  is minimized.

# Unconstrained Optimization



$F'(x) = 0$  and  $F''(x) < 0$  when  $x = 5$ .  $F(x)$  is maximized.

# Unconstrained Optimization



$F'(x) = 0$  and  $F''(x) > 0$  when  $x = 5$ .  $F(x)$  is minimized.



# Unconstrained Optimization

Theorem If  $x^*$  solves

$$\max_x F(x),$$

then  $x^*$  is a **critical point** of  $F$ , that is,

$$F'(x^*) = 0.$$

# Unconstrained Optimization

Theorem If

$$F'(x^*) = 0 \text{ and } F''(x^*) < 0,$$

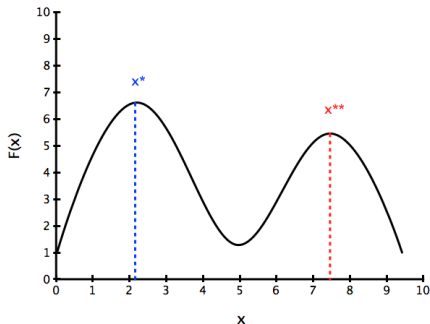
then  $x^*$  solves

$$\max_x F(x)$$

(at least locally).

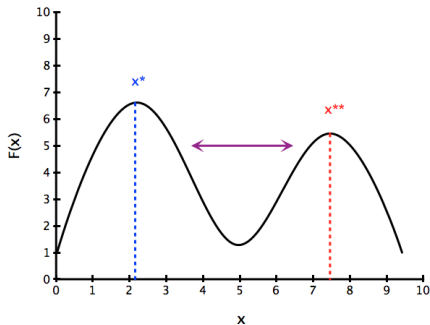
The first-order condition  $F'(x^*) = 0$  and the **second-order condition**  $F''(x^*) < 0$  are **sufficient** conditions for the value of  $x$  that (locally) maximizes  $F(x)$ .

# Unconstrained Optimization



$F'(x^{**}) = 0$  and  $F''(x^{**}) < 0$  at the **local** maximizer  $x^{**}$  and  
 $F'(x^*) = 0$  and  $F''(x^*) < 0$  at the **global** maximizer  $x^*$ .

# Unconstrained Optimization



$F'(x^{**}) = 0$  and  $F''(x^{**}) < 0$  at the local maximizer  $x^{**}$  and  $F'(x^*) = 0$  and  $F''(x^*) < 0$  at the global maximizer  $x^*$ , but  $F''(x) > 0$  in between  $x^*$  and  $x^{**}$ .

# Unconstrained Optimization

Theorem If

$$F'(x^*) = 0$$

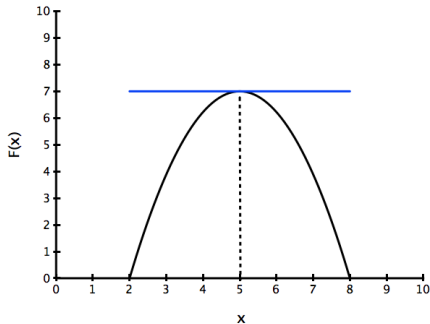
and

$$F''(x) < 0 \text{ for all } x \in \mathbb{R},$$

then  $x^*$  solves

$$\max_x F(x).$$

# Unconstrained Optimization



$F''(x) < 0$  for all  $x \in \mathbb{R}$  and  $F'(5) = 0$ .  $F(x)$  is maximized when  $x = 5$ .

# Unconstrained Optimization

If  $F''(x) < 0$  for all  $x \in \mathbb{R}$ , then the function  $F$  is **concave**.

When  $F$  is concave, the first-order condition  $F'(x^*) = 0$  is **both necessary and sufficient** for the value of  $x$  that maximizes  $F(x)$ .

And, as we are about to see, concave functions arise frequently and naturally in economics and finance.

# Unconstrained Optimization

Recall that if  $f(x) = ax$ , then  $f'(x) = a$ .

If  $g(x) = ax + b$ , then  $g'(x) = a$ .

And if  $h(x) = x^c$ , then  $h'(x) = cx^{c-1}$ .



# Unconstrained Optimization: Example 1

Consider the problem

$$\max_x \left( -\frac{1}{2} \right) (x - \tau)^2,$$

where  $\tau$  is a number ( $\tau \in \mathbb{R}$ ) that we might call the “target.”

The first-order condition

$$-(x^* - \tau) = 0$$

leads us immediately to the solution:  $x^* = \tau$ .