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FINANCIAL ECONOMICS

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Deriving the CAPM

$$E(\tilde{r}_j) = r_f + \beta_j[E(\tilde{r}_M) - r_f]$$

This equation summarizes a very strong restriction.

The expected return on any asset j depends on

$$\beta_j = \sigma_{jM}/\sigma_M^2$$

and only on β_j.

Interpreting the CAPM

There are several complementary ways of interpreting this result.

All bring us back to the theme of diversification emphasized by MPT.

But all take us a step further, by emphasizing as well the distinction between idiosyncratic risk, which can be "diversified away," and aggregate risk, which cannot be diversified away.

Interpreting the CAPM

The first interpretation goes directly back to the MPT: a stock with low and especially negative σ_{jM} will be most useful for diversification.

But then all investors will want to hold that stock. In equilibrium, therefore, the stock's price will be high and, given future cash flows, its expected return will be low.

Therefore, stocks with low or negative betas will have low expected returns. Investors hold these stocks, despite their low expected returns, because of they are useful for diversification.

Interpreting the CAPM

Conversely, a stock with high, positive σ_{jM} will not be very useful for diversification.

In equilibrium, therefore, the stock will sell for a low price.

Therefore, stocks with high betas will have high expected returns. The high expected return is needed to compensate investors, because the stock is not very useful for diversification.

Interpreting the CAPM

The second interpretation uses the CAPM equation in its original form

$$E(\tilde{r}_j) = r_f + \left(\frac{\sigma_{jM}}{\sigma_M^2}\right)[E(\tilde{r}_M) - r_f]$$

together with the definition of correlation, which implies

$$\rho_{jM} = \frac{\sigma_{jM}}{\sigma_j \sigma_M}$$

to re-express the CAPM relationship as

$$E(\tilde{r}_j) = r_f + \left[\frac{E(\tilde{r}_M) - r_f}{\sigma_M}\right]\rho_{jM}\sigma_j$$

Interpreting the CAPM

$$E(\tilde{r}_j) = r_f + \left[\frac{E(\tilde{r}_M) - r_f}{\sigma_M}\right] \rho_{jM}\sigma_j$$

The term inside brackets is the equilibrium price of risk.

And since the correlation lies between -1 and 1, the term $\rho_{jM}\sigma_j$, satisfying

$$\rho_{jM}\sigma_j \leq \sigma_j,$$

represents the "portion" of the total risk σ_j in asset j that is correlated with the market return.

Interpreting the CAPM

$$E(\tilde{r}_j) = r_f + \left[\frac{E(\tilde{r}_M) - r_f}{\sigma_M} \right] \rho_{jM} \sigma_j$$

The idiosyncratic risk in asset j, that is, the portion that is uncorrelated with the market return, can be diversified away by holding the market portfolio.

Since this risk can be freely shed through diversification, it is not "priced."

Interpreting the CAPM

$$E(\tilde{r}_j) = r_f + \left[\frac{E(\tilde{r}_M) - r_f}{\sigma_M}\right] \rho_{jM}\sigma_j$$

Hence, according to the CAPM, risk in asset j is priced only to the extent that it takes the form of aggregate risk that, because it is correlated with the market portfolio, cannot be diversified away.

Interpreting the CAPM

$$E(\tilde{r}_j) = r_f + \left[\frac{E(\tilde{r}_M) - r_f}{\sigma_M} \right] \rho_{jM} \sigma_j$$

Thus, according to the CAPM:

1. Only assets with random returns that are positively correlated with the market return earn expected returns above the risk free rate. They must, in order to induce investors to take on more aggregate risk.

Interpreting the CAPM

$$E(\tilde{r}_j) = r_f + \left[\frac{E(\tilde{r}_M) - r_f}{\sigma_M} \right] \rho_{jM} \sigma_j$$

Thus, according to the CAPM:

2. Assets with returns that are uncorrelated with the market return have expected returns equal to the risk free rate, since their risk can be completely diversified away.

Interpreting the CAPM

$$E(\tilde{r}_j) = r_f + \left[\frac{E(\tilde{r}_M) - r_f}{\sigma_M}\right] \rho_{jM}\sigma_j$$

Thus, according to the CAPM:

3. Assets with negative betas – that is, with random returns that are negatively correlated with the market return – have expected returns below the risk free rate! For these assets, $E(\tilde{r}_j) - r_f < 0$ is like an "insurance premium" that investors will pay in order to insulate themselves from aggregate risk.

Interpreting the CAPM

The third interpretation is based on a statistical regression of the random return $\tilde{r}_j$ on asset j on a constant and the market return $\tilde{r}_M$:

$$\tilde{r}_j = \alpha + \beta_j \tilde{r}_M + \varepsilon_j$$

This regression breaks the variance of $\tilde{r}_j$ down into two "orthogonal" (uncorrelated) components:

1. The component $\beta_j \tilde{r}_M$ that is systematically related to variation in the market return.
2. The component ε_j that is not.

Do you remember the formula for β_j, the slope coefficient in a linear regression?

Interpreting the CAPM

Consider a statistical regression of the random return $\tilde{r}_j$ on asset j on a constant and the market return $\tilde{r}_M$:

$$\tilde{r}_j = \alpha + \beta_j \tilde{r}_M + \varepsilon_j$$

Do you remember the formula for β_j, the slope coefficient in a linear regression? It is

$$\beta_j = \frac{\sigma_{jM}}{\sigma_M^2}$$

the same "beta" as in the CAPM!

Interpreting the CAPM

Consider a statistical regression:

$$\tilde{r}_j = \alpha + \beta_j \tilde{r}_M + \varepsilon_j \text{ with } \beta_j = \sigma_{jM}/\sigma_M^2$$

the same "beta" as in the CAPM!

But this is not an accident: to the contrary, it restates the conclusion that, according to the CAPM, risk in an individual asset is priced – and thereby reflected in a higher expected return – only to the extent that it is correlated with the market return.

Valuing Risky Cash Flows

We can also use the CAPM to value risky cash flows.

Let $\tilde{C}_{t+1}$ denote a random payoff to be received at time $t+1$ ("one period from now") and let P_t^C denote its price at time t ("today.")

If $\tilde{C}_{t+1}$ was known in advance, that is, if the payoff were riskless, we could find its value by discounting it at the risk free rate:

$$P_t^C = \frac{\tilde{C}_{t+1}}{1+r_f}$$

Valuing Risky Cash Flows

But when $\tilde{C}_{t+1}$ is truly random, we need to find its expected value $E(\tilde{C}_{t+1})$ and then "penalize" it for its riskiness either by discounting at a higher rate

$$P_t^C = \frac{E(\tilde{C}_{t+1})}{1 + r_f + \psi}$$

The CAPM can help us identify the appropriate risk premium ψ.

Valuing Risky Cash Flows

$$P_t^C = \frac{E(\tilde{C}_{t+1})}{1 + r_f + \psi}$$

Our previous analysis suggests that, broadly speaking, the risk premium ψ implied by the CAPM will somehow depend on the extent to which the random payoff $\tilde{C}_{t+1}$ is correlated with the return on the market portfolio.

Valuing Risky Cash Flows

To apply the CAPM to this valuation problem, we can start by observing that with price P_t^C today and random payoff $\tilde{C}_{t+1}$ one period from now, the return on this asset or investment project is defined by

$$1 + \tilde{r}_C = \frac{\tilde{C}_{t+1}}{P_t^C}$$

or

$$\tilde{r}_C = \frac{\tilde{C}_{t+1} - P_t^C}{P_t^C}$$

where the notation $\tilde{r}_C$ emphasizes that this return, like the future cash flow itself, is risky.

Valuing Risky Cash Flows

Now the CAPM implies that the expected return $E(\tilde{r}_C)$ must satisfy

$$E(\tilde{r}_C) = r_f + \beta_C[E(\tilde{r}_M) - r_f]$$

where the project's beta depends on the covariance of its return with the market return:

$$\beta_C = \frac{\sigma_{CM}}{\sigma_M^2}$$

This is what takes skill: with an existing asset, one can use data on the past correlation between its return and the market return to estimate beta. With a totally new project that is just being planned, a combination of experience, creativity, and hard work is often needed to choose the right value for β_C.

Valuing Risky Cash Flows

But once a value for β_C is determined, we can use

$$E(\tilde{r}_C) = r_f + \beta_C[E(\tilde{r}_M) - r_f]$$

together with the definition of the return itself

$$\tilde{r}_C = \frac{\tilde{C}_{t+1}}{P_t^C} - 1$$

to write

$$E\left(\frac{\tilde{C}_{t+1}}{P_t^C} - 1\right) = r_f + \beta_C[E(\tilde{r}_M) - r_f]$$

Valuing Risky Cash Flows

$$E\left(\frac{\tilde{C}_{t+1}}{P_t^C} - 1\right) = r_f + \beta_C[E(\tilde{r}_M) - r_f]$$

implies

$$\left(\frac{1}{P_t^C}\right) E(\tilde{C}_{t+1}) = 1 + r_f + \beta_C[E(\tilde{r}_M) - r_f]$$

$$P_t^C = \frac{E(\tilde{C}_{t+1})}{1 + r_f + \beta_C[E(\tilde{r}_M) - r_f]}$$

Valuing Risky Cash Flows

Hence, through

$$P_t^C = \frac{E(\tilde{C}_{t+1})}{1 + r_f + \beta_C[E(\tilde{r}_M) - r_f]}$$

the CAPM implies a risk premium of

$$\psi = \beta_C[E(\tilde{r}_M) - r_f]$$

which, as expected, depends critically on the covariance between the return on the risky project and the return on the market portfolio.