ECON 337901
FINANCIAL ECONOMICS

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Deriving the CAPM

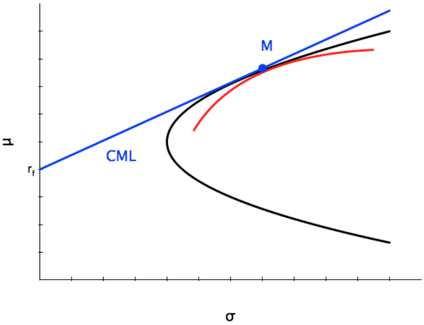
An investor who allocates share w to asset j and the remaining share $1 - w$ to the market as a whole will have a portfolio with

$$E(\tilde{r}_P) = wE(\tilde{r}_j) + (1 - w)E(\tilde{r}_M),$$

$$\sigma_P^2 = w^2\sigma_j^2 + (1 - w)^2\sigma_M^2 + 2w(1 - w)\sigma_{jM},$$

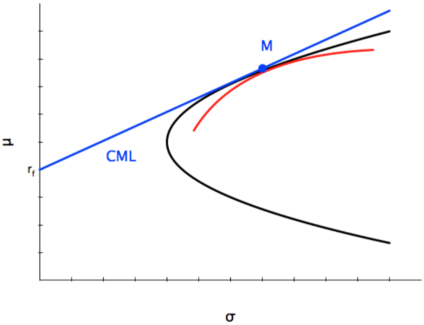
We can use these formulas to trace out how σ_P and $E(\tilde{r}_P)$ vary as w changes. Note that once again, covariance is going to matter.

Deriving the CAPM



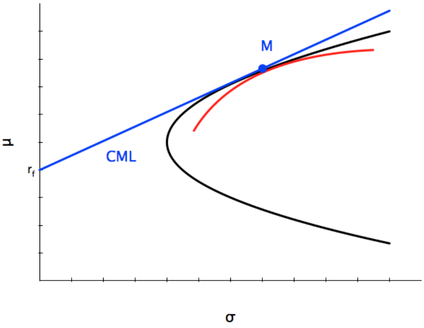
The red curve these traces out how σ_P and $E(\tilde{r}_P)$ vary as w changes, that is, as asset j gets underweighted or overweighted relative to the market portfolio.

Deriving the CAPM



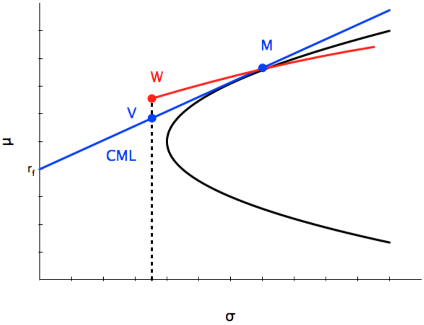
The red curve passes through M, since when $w = 0$ the new portfolio coincides with the market portfolio.

Deriving the CAPM



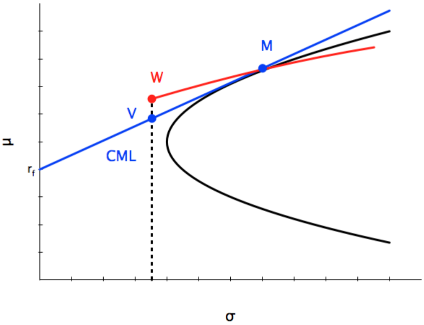
For all other values of w, however, the red curve must lie below the CML.

Deriving the CAPM



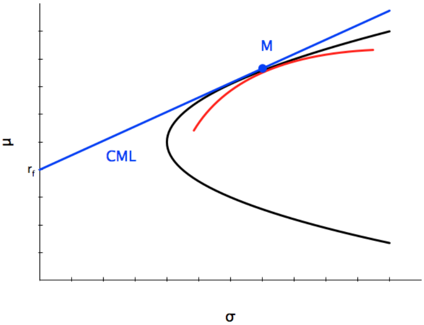
Otherwise, a portfolio along the CML would be dominated in mean-variance by the new portfolio. Financial markets would no longer be in equilibrium, since some investors would no longer be willing to hold the market portfolio.

Deriving the CAPM



Investors as a group cannot overweight or underweight asset j in their portfolios. Asset prices will have to adjust so that, in equilibrium, "everyone owns the market."

Deriving the CAPM



Together, these observations imply that the red curve must be tangent to the CML at M.

Deriving the CAPM

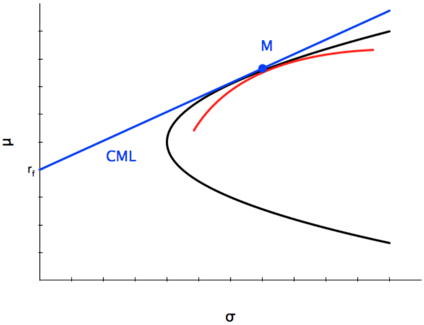
Tangent means equal in slope.

We already know that the slope of the Capital Market Line is

$$\frac{E(\tilde{r}_M) - r_f}{\sigma_M}$$

But what is the slope of the red curve?

Deriving the CAPM



Let $f(\sigma_P)$ be the function defined by $E(\tilde{r}_P) = f(\sigma_P)$ and therefore describing the red curve.

Deriving the CAPM

Next, define the functions $g(w)$ and $h(w)$ by

$$g(w) = wE(\tilde{r}_j) + (1-w)E(\tilde{r}_M),$$

$$h(w) = [w^2\sigma_j^2 + (1-w)^2\sigma_M^2 + 2w(1-w)\sigma_{jM}]^{1/2},$$

so that

$$E(\tilde{r}_P) = g(w)$$

and

$$\sigma_P = h(w).$$

Deriving the CAPM

Substitute

$$E(\tilde{r}_P) = g(w)$$

and

$$\sigma_P = h(w).$$

into

$$E(\tilde{r}_P) = f(\sigma_P)$$

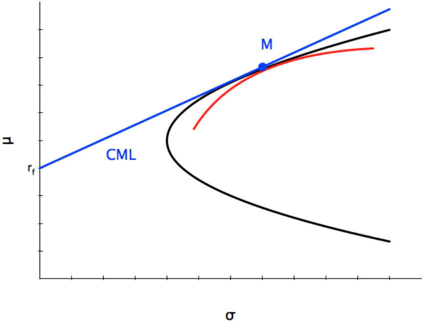
to obtain

$$g(w) = f(h(w))$$

and use the chain rule to compute

$$g'(w) = f'(h(w))h'(w) = f'(\sigma_P)h'(w)$$

Deriving the CAPM



Let $f(\sigma_P)$ be the function defined by $E(\tilde{r}_P) = f(\sigma_P)$ and therefore describing the red curve. Then $f'(\sigma_P)$ is the slope of the curve.

Deriving the CAPM

Hence, to compute $f'(\sigma_P)$, we can rearrange

$$g'(w) = f'(\sigma_P)h'(w)$$

to obtain

$$f'(\sigma_P) = \frac{g'(w)}{h'(w)}$$

and compute $g'(w)$ and $h'(w)$ from the formulas we know.

Deriving the CAPM

$$g(w) = wE(\tilde{r}_j) + (1 - w)E(\tilde{r}_M),$$

implies

$$g'(w) = E(\tilde{r}_j) - E(\tilde{r}_M)$$

Deriving the CAPM

$$h(w) = [w^2\sigma_j^2 + (1-w)^2\sigma_M^2 + 2w(1-w)\sigma_{jM}]^{1/2},$$

implies

$$h'(w) = \frac{1}{2}\left\{\frac{2w\sigma_j^2 - 2(1-w)\sigma_M^2 + 2(1-2w)\sigma_{jM}}{[w^2\sigma_j^2 + (1-w)^2\sigma_M^2 + 2w(1-w)\sigma_{jM}]^{1/2}}\right\}$$

or, a bit more simply,

$$h'(w) = \frac{w\sigma_j^2 - (1-w)\sigma_M^2 + (1-2w)\sigma_{jM}}{[w^2\sigma_j^2 + (1-w)^2\sigma_M^2 + 2w(1-w)\sigma_{jM}]^{1/2}}$$

Deriving the CAPM

$$f'(\sigma_P) = g'(w)/h'(w)$$

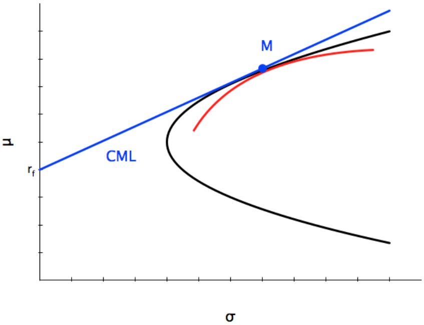
$$g'(w) = E(\tilde{r}_j) - E(\tilde{r}_M)$$

$$h'(w) = \frac{w\sigma_j^2 - (1-w)\sigma_M^2 + (1-2w)\sigma_{jM}}{[w^2\sigma_j^2 + (1-w)^2\sigma_M^2 + 2w(1-w)\sigma_{jM}]^{1/2}}$$

imply

$$\begin{aligned} f'(\sigma_P) &= [E(\tilde{r}_j) - E(\tilde{r}_M)] \\ &\quad \times \frac{[w^2\sigma_j^2 + (1-w)^2\sigma_M^2 + 2w(1-w)\sigma_{jM}]^{1/2}}{w\sigma_j^2 - (1-w)\sigma_M^2 + (1-2w)\sigma_{jM}} \end{aligned}$$

Deriving the CAPM



The red curve is tangent to the CML at M. Hence, $f'(\sigma_P)$ equals the slope of the CML when w=0.

Deriving the CAPM

When $w = 0$,

$$\begin{aligned} f'(\sigma_P) &= [E(\tilde{r}_j) - E(\tilde{r}_M)] \\ &\quad \times \frac{[w^2\sigma_j^2 + (1-w)^2\sigma_M^2 + 2w(1-w)\sigma_{jM}]^{1/2}}{w\sigma_j^2 - (1-w)\sigma_M^2 + (1-2w)\sigma_{jM}} \end{aligned}$$

implies

$$f'(\sigma_P) = \frac{[E(\tilde{r}_j) - E(\tilde{r}_M)]\sigma_M}{\sigma_{jM} - \sigma_M^2}$$

Meanwhile, we know that the slope of the CML is

$$\frac{E(\tilde{r}_M) - r_f}{\sigma_M}$$

Deriving the CAPM

The tangency of the red curve with the CML at M therefore requires

$$\frac{[E(\tilde{r}_j) - E(\tilde{r}_M)]\sigma_M}{\sigma_{jM} - \sigma_M^2} = \frac{E(\tilde{r}_M) - r_f}{\sigma_M}$$

$$E(\tilde{r}_j) - E(\tilde{r}_M) = \frac{[E(\tilde{r}_M) - r_f][\sigma_{jM} - \sigma_M^2]}{\sigma_M^2}$$

$$E(\tilde{r}_j) - E(\tilde{r}_M) = \left(\frac{\sigma_{jM}}{\sigma_M^2}\right)[E(\tilde{r}_M) - r_f] - [E(\tilde{r}_M) - r_f]$$

$$E(\tilde{r}_j) = r_f + \left(\frac{\sigma_{jM}}{\sigma_M^2}\right)[E(\tilde{r}_M) - r_f]$$

Deriving the CAPM

$$E(\tilde{r}_j) = r_f + \left(\frac{\sigma_{jM}}{\sigma_M^2}\right)[E(\tilde{r}_M) - r_f]$$

Let

$$\beta_j = \frac{\sigma_{jM}}{\sigma_M^2}$$

so that this key equation of the CAPM can be written as

$$E(\tilde{r}_j) = r_f + \beta_j[E(\tilde{r}_M) - r_f]$$

where β_j, the "beta" for asset j, depends on the covariance between the returns on asset j and the market portfolio.

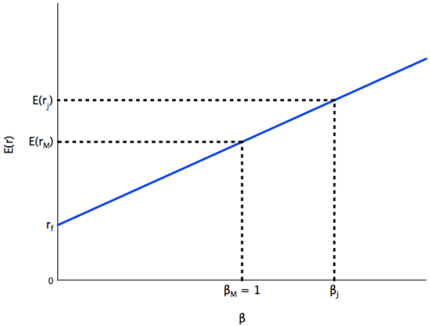
Deriving the CAPM

$$E(\tilde{r}_j) = r_f + \beta_j[E(\tilde{r}_M) - r_f]$$

This equation summarizes a very strong restriction.

It implies that if we rank individual stocks or portfolios of stocks according to their betas, their expected returns should all lie along a single security market line with slope $E(\tilde{r}_M) - r_f$.

Deriving the CAPM



According to the CAPM, all assets and portfolios of assets lie along a single security market line. Those with higher betas have higher expected returns.

Testing the CAPM

An enormous literature is devoted to empirically testing the CAPM's implications.

Although results are mixed, studies have shown that when individual portfolios are ranked according to their betas, expected returns tend to line up as suggested by the theory.

Testing the CAPM

A famous article that presents results along these lines is by Eugene Fama (Nobel Prize 2013) and James MacBeth, "Risk, Return, and Equilibrium," *Journal of Political Economy* Vol.81 (May-June 1973), pp.607-636.

Early work on the MPT, the CAPM, and econometric tests of the efficient markets hypothesis and the CAPM is discussed extensively in Eugene Fama's 1976 textbook, *Foundations of Finance*.

Testing the CAPM

More recent evidence against the CAPM's implications is presented by Eugene Fama and Kenneth French, "Common Risk Factors in the Returns on Stocks and Bonds," *Journal of Financial Economics* Vol.33 (February 1993): pp.3-56.

This paper shows that equity shares in small firms and in firms with high book (accounting) to market value have expected returns that differ strongly from what is predicted by the CAPM alone.

Testing the CAPM

Quite a bit of recent research has been directed towards understanding the source of these "anomalies."

See, for example, David McLean and Jeffrey Pontiff, "Does Academic Research Destroy Stock Return Predictability?" *Journal of Finance* Vol.71 (February 2016): pp.5-31.