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FINANCIAL ECONOMICS

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Portfolios, Risk Aversion, and Wealth

The problem

$$\max_a \quad \pi \left\{ \frac{[Y_0(1+r_f)+a(r_G-r_f)]^{1-\gamma}-1}{1-\gamma} \right\}$$
$$+ (1-\pi) \left\{ \frac{[Y_0(1+r_f)+a(r_B-r_f)]^{1-\gamma}-1}{1-\gamma} \right\}$$

has first-order condition

$$\frac{\pi(r_G-r_f)}{[Y_0(1+r_f)+a^*(r_G-r_f)]^{\gamma}} + \frac{(1-\pi)(r_B-r_f)}{[Y_0(1+r_f)+a^*(r_B-r_f)]^{\gamma}} = 0.$$

Portfolios, Risk Aversion, and Wealth

$$\frac{a^*}{Y_0} = \frac{(1+r_f)\{[\pi(r_G - r_f)]^{1/\gamma} - [(1-\pi)(r_f - r_B)]^{1/\gamma}]\}}{(r_G - r_f)[(1-\pi)(r_f - r_B)]^{1/\gamma} + (r_f - r_B)[\pi(r_G - r_f)]^{1/\gamma}}$$

γ	r_f	r_G	r_B	π	$E(\tilde{r})$	a^*/Y_0
0.5	0.05	0.40	−0.20	0.50	0.10	1.20
1	0.05	0.40	−0.20	0.50	0.10	0.60
2	0.05	0.40	−0.20	0.50	0.10	0.30
3	0.05	0.40	−0.20	0.50	0.10	0.20
5	0.05	0.40	−0.20	0.50	0.10	0.12
10	0.05	0.40	−0.20	0.50	0.10	0.06

Portfolios, Risk Aversion, and Wealth

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Consistent with Arrow's theorem, higher coefficients of relative risk aversion are associated with smaller values of a^*.

Portfolios, Risk Aversion, and Wealth

$$\frac{a^*}{Y_0} = \frac{(1+r_f)\{[\pi(r_G - r_f)]^{1/\gamma} - [(1-\pi)(r_f - r_B)]^{1/\gamma}]\}}{(r_G - r_f)[(1-\pi)(r_f - r_B)]^{1/\gamma} + (r_f - r_B)[\pi(r_G - r_f)]^{1/\gamma}}$$

Also consistent with Arrow's results, we see here that with constant relative risk aversion, a^* rises proportionally with wealth.

6 Modern Portfolio Theory

A Generalizing the Portfolio Problem

B Justifying Mean-Variance Utility

C The Gains From Diversification

D The Efficient Frontier

E A Separation Theorem

F Strengths and Shortcomings of MPT

Generalizing the Portfolio Problem

We can elaborate on our previous portfolio problem

$$\max_a E\{u[Y_0(1 + r_f) + a(\tilde{r} - r_f)]\}$$

by allowing the investor to allocate funds to $N > 1$ risky assets.

$\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_N$	risky (random) returns
$a_1, a_2, \ldots, a_N$	amounts allocated at the risky assets
$w_i = a_i/Y_0$	share of initial wealth allocated to each risky asset (portfolio *w*eights)

Generalizing the Portfolio Problem

$$\begin{aligned}
\tilde{Y}_1 &= \text{random terminal wealth} \\
&= (1 + r_f)\left(Y_0 - \sum_{i=1}^{N} a_i\right) + \sum_{i=1}^{N} a_i(1 + \tilde{r}_i) \\
&= (1 + r_f)Y_0 + \sum_{i=1}^{N} a_i(\tilde{r}_i - r_f) \\
&= (1 + r_f)Y_0 + \sum_{i=1}^{N} w_i Y_0(\tilde{r}_i - r_f)
\end{aligned}$$

Generalizing the Portfolio Problem

With

$$\tilde{Y}_1 = (1 + r_f)Y_0 + \sum_{i=1}^{N} w_i Y_0(\tilde{r}_i - r_f),$$

the generalized problem can be stated as

$$\max_{w_1, w_2, \ldots, w_N} E\left\{u\left[Y_0(1 + r_f) + \sum_{i=1}^{N} w_i Y_0(\tilde{r}_i - r_f)\right]\right\}$$

Generalizing the Portfolio Problem

Modern Portfolio Theory examines the solution to this extended problem assuming that investors have mean-variance utility, that is, assuming that investors' preferences can be represented by a trade-off between the mean (expected value) and variance (or standard deviation) of terminal wealth.

MPT was developed by Harry Markowitz (US, b.1927, Nobel Prize 1990) in the early 1950s, the classic paper being his article "Portfolio Selection," *Journal of Finance* Vol.7 (March 1952): pp.77-91.

Justifying Mean-Variance Utility

The mean-variance utility hypothesis seemed natural at the time the MPT first appeared, and it retains some intuitive appeal today. But viewed in the context of more recent developments in financial economics, particularly the development of vN-M expected utility theory, it now looks a bit peculiar.

A first question for us, therefore, is: Under what conditions will investors have preferences over the means and variances of asset returns?

Justifying Mean-Variance Utility

Under what conditions will investors have preferences over the means and variances of asset returns?

There are three arguments:

1. Quadratic approximation to a general Bernoulli utility function.
2. Quadratic Bernoulli utility function.
3. Asset returns are normally distributed.

Justifying Mean-Variance Utility

The first argument uses a quadratic approximation:

$$f(x+a) \approx f(x) + f'(x)a + \frac{1}{2}f''(x)a^2$$

"Decompose" terminal wealth $\tilde{Y}_1$ as

$$\tilde{Y}_1 = E(\tilde{Y}_1) + [\tilde{Y}_1 - E(\tilde{Y}_1)]$$

where $x + a = \tilde{Y}_1$, $x = E(\tilde{Y}_1)$, and

$$a = \tilde{Y}_1 - E(\tilde{Y}_1)$$

is the size of the "bet."

Justifying Mean-Variance Utility

With $x + a = \tilde{Y}_1$, $x = E(\tilde{Y}_1)$, and $a = \tilde{Y}_1 - E(\tilde{Y}_1)$,

$$f(x+a) \approx f(x) + f'(x)a + \frac{1}{2}f''(x)a^2$$

$$\begin{aligned} u(\tilde{Y}_1) &\approx u[E(\tilde{Y}_1)] + u'[E(\tilde{Y}_1)][\tilde{Y}_1 - E(\tilde{Y}_1)] \\ &\quad + \frac{1}{2}u''[E(\tilde{Y}_1)][\tilde{Y}_1 - E(\tilde{Y}_1)]^2 \end{aligned}$$

Justifying Mean-Variance Utility

If $\tilde{X}$ is random and a is known, then $E(a\tilde{X}) = aE(\tilde{X})$.

Therefore

$$\begin{aligned} u(\tilde{Y}_1) &\approx u[E(\tilde{Y}_1)] + u'[E(\tilde{Y}_1)][\tilde{Y}_1 - E(\tilde{Y}_1)] \\ &+ \frac{1}{2}u''[E(\tilde{Y}_1)][\tilde{Y}_1 - E(\tilde{Y}_1)]^2 \end{aligned}$$

implies that an approximation to expected utility is

$$\begin{aligned} E[u(\tilde{Y}_1)] &\approx u[E(\tilde{Y}_1)] + u'[E(\tilde{Y}_1)]E[\tilde{Y}_1 - E(\tilde{Y}_1)] \\ &+ \frac{1}{2}u''[E(\tilde{Y}_1)]E[\tilde{Y}_1 - E(\tilde{Y}_1)]^2 \end{aligned}$$

Justifying Mean-Variance Utility

Since

$$E[\tilde{Y}_1 - E(\tilde{Y}_1)] = 0 \text{ and } E[\tilde{Y}_1 - E(\tilde{Y}_1)]^2 = \sigma^2(\tilde{Y}_1)$$

$$\begin{aligned} E[u(\tilde{Y}_1)] &\approx u[E(\tilde{Y}_1)] + u'[E(\tilde{Y}_1)]E[\tilde{Y}_1 - E(\tilde{Y}_1)] \\ &+ \frac{1}{2}u''[E(\tilde{Y}_1)]E[\tilde{Y}_1 - E(\tilde{Y}_1)]^2 \end{aligned}$$

simplifies to

$$E[u(\tilde{Y}_1)] \approx u[E(\tilde{Y}_1)] + \frac{1}{2}u''[E(\tilde{Y}_1)]\sigma^2(\tilde{Y}_1)$$

Justifying Mean-Variance Utility

$$E[u(\tilde{Y}_1)] \approx u[E(\tilde{Y}_1)] + \frac{1}{2}u''[E(\tilde{Y}_1)]\sigma^2(\tilde{Y}_1)$$

The right-hand side of this expression is in the desired form: if u is increasing, it rewards higher mean returns and if u is concave, it penalizes higher variance in returns.

So one possible justification for mean-variance utility is to assume that the size of the portfolio bet $\tilde{Y}_1 - E(\tilde{Y}_1)$ is small enough to make this Taylor approximation a good one.

But is it safe to assume that portfolio bets are small?

Justifying Mean-Variance Utility

A second argument is to assume that the Bernoulli utility function is quadratic, so that the quadratic approximation holds exactly, even for large bets.

With

$$u(Y) = a + bY + cY^2,$$

$$u'(Y) = b + 2cY$$

and

$$u''(Y) = 2c$$

mean that $c < 0$ for risk aversion and $b > 0$ if more is preferred to less.

Justifying Mean-Variance Utility

Note, however, that $u'(Y) = b + 2cY$ and $u''(Y) = 2c$ imply

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = -\frac{2c}{b + 2cY} = -2c(b + 2cY)^{-1}$$

and therefore

$$R_A'(Y) = 2c(b + 2cY)^{-2}(2c) = \left(\frac{2c}{b + 2cY}\right)^2 > 0$$

so that quadratic utility implies increasing absolute risk aversion.

Justifying Mean-Variance Utility

Fortunately, there is a third argument.

If all individual risky asset returns are normally distributed, then terminal wealth $\tilde{Y}_1$ will be normally distributed as well.

And if $\tilde{Y}_1$ is normally distributed with mean $\mu_Y = E(\tilde{Y}_1)$ and standard deviation $\sigma_Y = \{E[\tilde{Y}_1 - E(\tilde{Y}_1)]^2\}^{1/2}$ then the expectation of any function of $\tilde{Y}_1$ can be written as a function of μ_Y and σ_Y:

$$E[u(\tilde{Y}_1)] = v(\mu_Y, \sigma_Y)$$

Justifying Mean-Variance Utility

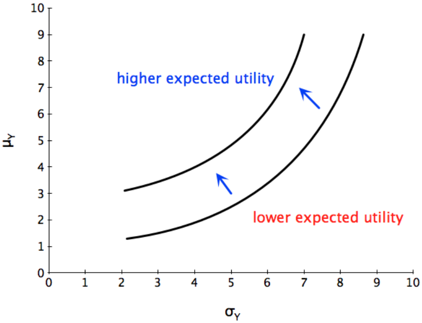
If $\tilde{Y}_1$ is normally distributed, there exists a function v such that

$$E[u(\tilde{Y}_1)] = v(\mu_Y, \sigma_Y).$$

Moreover, if $\tilde{Y}_1$ is normally distributed and

1. u is increasing, then v is increasing in μ_Y
2. u is concave, then v is decreasing in σ_Y
3. u is concave, then indifference curves defined over μ_Y and σ_Y are convex

Justifying Mean-Variance Utility



Since μ_Y is a "good" and σ_Y is a "bad," indifference curves slope up. But if u is concave, these indifference curves will still be convex.

Justifying Mean-Variance Utility

Returns on individual stocks and stock indices are approximately normal, but:

1. Returns on assets like options are highly non-normal.
2. Departures from normality, including skewness (asymmetry) and excess kurtosis ("fat tails"), can be detected in returns on individual stocks and the market as a whole.

Basically, stock market crashes happen more often than they would if returns were truly normal.

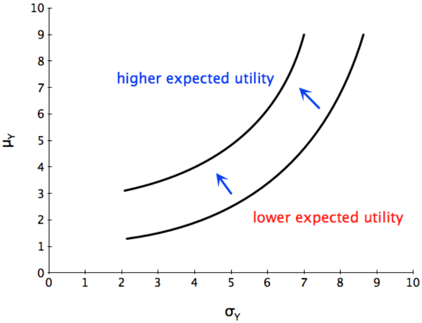
Justifying Mean-Variance Utility

The mean-variance utility hypothesis is intuitively appealing and can be justified with reference to vN-M expected utility theory by assuming risky asset returns are normally distributed.

That's why people say, "the CAPM requires normal returns."

It's also why people say, "the CAPM can't be used to price options."

Justifying Mean-Variance Utility



But what does the “budget constraint” look like in this diagram? To see, we need to consider the gains from diversification.