# ECON 337901
# FINANCIAL ECONOMICS

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# 5 Risk Aversion and Investment Decisions

A Risk Aversion and Portfolio Allocation

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# Risk Aversion and Portfolio Allocation

Let's now put our framework of decision-making under uncertainty to use.

Consider a risk-averse investor with vN-M expected utility who divides his or her initial wealth $Y_0$ into an amount $a$ allocated to a risky asset – say, the stock market – and an amount $Y_0 - a$ allocated to a safe asset – say, a bank account or a government bond.

# Risk Aversion and Portfolio Allocation

$Y_0$ = initial wealth

$a$ = amount allocated to stocks

$\tilde{r}$ = random return on stocks

$r_f$ = risk-free return

$\tilde{Y}_1$ = terminal wealth

$$\begin{aligned} \tilde{Y}_1 &= (1 + r_f)(Y_0 - a) + a(1 + \tilde{r}) \\ &= Y_0(1 + r_f) + a(\tilde{r} - r_f) \end{aligned}$$

# Risk Aversion and Portfolio Allocation

The investor chooses $a$ to maximize expected utility:

$$\max_a E[u(\tilde{Y}_1)] = \max_a E\{u[Y_0(1 + r_f) + a(\tilde{r} - r_f)]\}$$

The first-order condition is

$$E\{u'[Y_0(1 + r_f) + a^*(\tilde{r} - r_f)](\tilde{r} - r_f)\} = 0.$$

Note: we are allowing the investor to sell stocks short ($a^* < 0$) or to buy stocks on margin ($a^* > Y_0$) if he or she desires.

# Risk Aversion and Portfolio Allocation

The following results were proven by Kenneth Arrow in "The Theory of Risk Aversion," published in the 1971 volume *Essays in the Theory of Risk-Bearing* and reprinted in 1983 in volume 3 of the *Collected Papers of Kenneth J. Arrow* (Harvard University Press).

# Risk Aversion and Portfolio Allocation

Theorem If the Bernoulli utility function $u$ is increasing and concave, then

$$a^* > 0 \text{ if and only if } E(\tilde{r}) > r_f$$

$$a^* = 0 \text{ if and only if } E(\tilde{r}) = r_f$$

$$a^* < 0 \text{ if and only if } E(\tilde{r}) < r_f$$

Thus, a risk-averse investor will always allocate at least some funds to the stock market if the expected return on stocks exceeds the risk-free rate.

# Risk Aversion and Portfolio Allocation

Danthine and Donaldson (3rd ed., p.41) report that in the United States, 1889-2010, average real (inflation-adjusted) returns on stocks and risk-free bonds are

$$E(\tilde{r}) = 0.075 \text{ (7.5 percent per year)}$$

$$r_f = 0.011 \text{ (1.1 percent per year)}$$

The equity risk premium of $E(\tilde{r}) - r_f = 0.064$ (6.4 percent) is not only positive, it is huge. The implication of the theory is that all investors, even the most risk averse, should have some money invested in the stock market

# Risk Aversion and Portfolio Allocation

Using updated data (1871-2020) on the S&P 500, adjusted for dividends and inflation from Robert Shiller's webpage:

`http://www.econ.yale.edu/~shiller/data.htm`

$$E(\tilde{r}) = 0.085 \qquad \sigma(\tilde{r}) = 0.179$$

$$E(\tilde{r}) - 2\sigma(\tilde{r}) = -0.273 \qquad E(\tilde{r}) + 2\sigma(\tilde{r}) = 0.443$$

$$r_{1931} = -0.380 \qquad r_{1933} = 0.531$$

$$r_{2008} = -0.356 \qquad r_{2009} = 0.298$$

$$r_{2019} = 0.250 \qquad r_{2020} = 0.161$$

# Risk Aversion and Portfolio Allocation

Arrow also showed that $a^*$ rises when either $R_A(Y_0)$ or $R_R(Y_0)$ goes down.

If there are investors with the same initial wealth but different coefficients of risk aversion, the investor who is less risk averse will hold more stocks.

# Risk Aversion and Portfolio Allocation

Arrow also showed that if an investor has:

1. Decreasing absolute risk aversion, then $a^*$ rises with $Y_0$.
2. Constant absolute risk aversion, then $a^*$ does not depend on $Y_0$.
3. Increasing absolute risk aversion, then $a^*$ falls with $Y_0$.

Intuitively, if $R_A(Y_0)$ is constant (does not depend on $Y_0$), then the investor finds his or her "optimal absolute bet" $a^*$ and sticks with it, even as income goes up.

# Risk Aversion and Portfolio Allocation

Arrow also showed that if an investor has:

1. Decreasing relative risk aversion, then $a^*/Y_0$ rises with $Y_0$.
2. Constant relative risk aversion, then $a^*/Y_0$ does not depend on $Y_0$.
3. Increasing relative risk aversion, then $a^*/Y_0$ falls with $Y_0$.

Intuitively, if $R_R(Y_0)$ is constant (does not depend on $Y_0$), then the investor finds his or her "optimal bet" $a^*/Y_0$ as a fraction of income, and lets $a^*$ rise proportionally as his or her income goes up.

# Risk Aversion and Portfolio Allocation

As an example, suppose $u(Y) = \ln(Y)$, as suggested by Daniel Bernoulli. Recall that for this utility function, $u'(Y) = 1/Y$. Then assume that stock returns can either be good or bad:

$$\tilde{r} = \begin{cases} r_G & \text{with probability } \pi \\ r_B & \text{with probability } 1 - \pi \end{cases}$$

where $r_G > r_f > r_B$ defines the "good" and "bad" states and

$$\pi r_G + (1 - \pi) r_B > r_f,$$

so that $E(\tilde{r}) > r_f$ and the investor will choose $a^* > 0$.

# Risk Aversion and Portfolio Allocation

The problem

$$\max_a E\{u[Y_0(1+r_f)+a(\tilde{r}-r_f)]\}$$

specializes to

$$\max_a \quad \pi \ln[Y_0(1+r_f)+a(r_G-r_f)] + (1-\pi)\ln[Y_0(1+r_f)+a(r_B-r_f)]$$

# Risk Aversion and Portfolio Allocation

Problem Set 10, Question 1: Specialize the problem

$$\max_a \quad \pi \ln[Y_0(1+r_f) + a(r_G - r_f)] + (1-\pi)\ln[Y_0(1+r_f) + a(r_B - r_f)]$$

by setting $Y_0 = 100$, $r_f = 0.10$, $r_G = 0.30$, $r_B = 0.05$, and $\pi = 1 - \pi = 1/2$.

The problem becomes

$$\max_a \, (1/2)\ln(110 + 0.20a) + (1/2)\ln(110 - 0.05a)$$

To find $a^*$, differentiate with respect to $a$ using the chain rule and set the result equal to zero.

# Risk Aversion and Portfolio Allocation

The problem

$$\max_a \quad \pi \ln[Y_0(1+r_f) + a(r_G - r_f)] + (1-\pi)\ln[Y_0(1+r_f) + a(r_B - r_f)]$$

has first-order condition

$$\frac{\pi(r_G - r_f)}{Y_0(1+r_f) + a^*(r_G - r_f)} + \frac{(1-\pi)(r_B - r_f)}{Y_0(1+r_f) + a^*(r_B - r_f)} = 0.$$

# Risk Aversion and Portfolio Allocation

$$\frac{\pi(r_G - r_f)}{Y_0(1 + r_f) + a^*(r_G - r_f)} + \frac{(1-\pi)(r_B - r_f)}{Y_0(1 + r_f) + a^*(r_B - r_f)} = 0$$

$$\begin{aligned} & \pi(r_G - r_f)[Y_0(1 + r_f) + a^*(r_B - r_f)] \\ = \; & -(1-\pi)(r_B - r_f)[Y_0(1 + r_f) + a^*(r_G - r_f)] \end{aligned}$$

$$\begin{aligned} & a^*(r_G - r_f)(r_B - r_f) \\ = \; & -Y_0(1 + r_f)[\pi(r_G - r_f) + (1-\pi)(r_B - r_f)] \end{aligned}$$

# Risk Aversion and Portfolio Allocation

$$\begin{aligned} & a^*(r_G - r_f)(r_B - r_f) \\ = \; & -Y_0(1+r_f)[\pi(r_G - r_f) + (1-\pi)(r_B - r_f)] \end{aligned}$$

implies

$$\frac{a^*}{Y_0} = -\frac{(1+r_f)[\pi(r_G - r_f) + (1-\pi)(r_B - r_f)]}{(r_G - r_f)(r_B - r_f)},$$

which is positive, since $r_G > r_f > r_B$ and

$$E(\tilde{r}) - r_f = \pi(r_G - r_f) + (1-\pi)(r_B - r_f) > 0.$$

# Risk Aversion and Portfolio Allocation

$$\frac{a^*}{Y_0} = -\frac{(1+r_f)[\pi(r_G - r_f) + (1-\pi)(r_B - r_f)]}{(r_G - r_f)(r_B - r_f)},$$

In this case, $a^*$:

Rises proportionally with $Y_0$.

Increases as $E(\tilde{r}) - r_f$ rises.

Falls as $r_G$ and $r_B$ move father away from $r_f$, holding $E(\tilde{r})$ constant; that is, in response to a "mean preserving spread."

# Risk Aversion and Portfolio Allocation

$$\frac{a^*}{Y_0} = -\frac{(1+r_f)[\pi(r_G - r_f) + (1-\pi)(r_B - r_f)]}{(r_G - r_f)(r_B - r_f)},$$

| $r_f$ | $r_G$ | $r_B$ | $\pi$ | $E(\tilde{r})$ | $a^*/Y_0$ |
|---|---|---|---|---|---|
| 0.05 | 0.40 | −0.20 | 0.50 | 0.10 | 0.60 |
| 0.05 | 0.30 | −0.10 | 0.50 | 0.10 | 1.40 |
| 0.05 | 0.40 | −0.20 | 0.75 | 0.25 | 2.40 |

The fraction of initial wealth allocated to stocks rises when stocks become less risky or pay higher expected returns.

# Portfolios, Risk Aversion, and Wealth

Let's generalize our previous example with logarithmic utility to the case where

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma},$$

with $\gamma > 0$. For this Bernoulli utility function, the coefficient of relative risk aversion is constant and equal to $\gamma$. The specific setting $\gamma = 1$ takes us back to the case with logarithmic utility.

# Portfolios, Risk Aversion, and Wealth

Hence, in this extended example,

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1-\gamma} \text{ implies } u'(Y) = Y^{-\gamma} = \frac{1}{Y^{\gamma}}.$$

and stock returns can either be good or bad

$$\tilde{r} = \begin{cases} r_G & \text{with probability } \pi \\ r_B & \text{with probability } 1-\pi \end{cases}$$

where $r_G > r_f > r_B$ defines the "good" and "bad" states and

$$\pi r_G + (1-\pi) r_B > r_f,$$

so that $E(\tilde{r}) > r_f$ and the investor will choose $a^* > 0$.

# Portfolios, Risk Aversion, and Wealth

With CRRA (constant relative risk aversion) utility and two states for $\tilde{r}$, the problem

$$\max_a E\{u[Y_0(1+r_f)+a(\tilde{r}-r_f)]\}$$

specializes to

$$\max_a \quad \pi\left\{\frac{[Y_0(1+r_f)+a(r_G-r_f)]^{1-\gamma}-1}{1-\gamma}\right\} + (1-\pi)\left\{\frac{[Y_0(1+r_f)+a(r_B-r_f)]^{1-\gamma}-1}{1-\gamma}\right\}$$

# Portfolios, Risk Aversion, and Wealth

The problem

$$\max_{a} \quad \pi \left\{ \frac{[Y_0(1+r_f) + a(r_G - r_f)]^{1-\gamma} - 1}{1-\gamma} \right\} + (1-\pi) \left\{ \frac{[Y_0(1+r_f) + a(r_B - r_f)]^{1-\gamma} - 1}{1-\gamma} \right\}$$

has first-order condition

$$\frac{\pi(r_G - r_f)}{[Y_0(1+r_f) + a^*(r_G - r_f)]^{\gamma}} + \frac{(1-\pi)(r_B - r_f)}{[Y_0(1+r_f) + a^*(r_B - r_f)]^{\gamma}} = 0.$$

# Portfolios, Risk Aversion, and Wealth

$$\frac{\pi(r_G - r_f)}{[Y_0(1+r_f) + a^*(r_G - r_f)]^\gamma} + \frac{(1-\pi)(r_B - r_f)}{[Y_0(1+r_f) + a^*(r_B - r_f)]^\gamma} = 0.$$

Notice that with $\gamma = 1$, this first-order condition specializes to the one that we derived last time, assuming log utility.

# Portfolios, Risk Aversion, and Wealth

$$\frac{\pi(r_G - r_f)}{[Y_0(1+r_f) + a^*(r_G - r_f)]^\gamma} + \frac{(1-\pi)(r_B - r_f)}{[Y_0(1+r_f) + a^*(r_B - r_f)]^\gamma} = 0.$$

Problem Set 10, Question 2: Set $Y_0 = 100$, $r_G = 0.30$, $r_B = 0.05$, $r_f = 0.10$, $\pi = 1 - \pi = 1/2$, and $\gamma = 2$. Since the constant coefficient of relative risk aversion goes up to 2 from 1, we should expect to see $a^*$ decline, relative to question 1.

# Portfolios, Risk Aversion, and Wealth

More generally,

$$\frac{\pi(r_G - r_f)}{[Y_0(1+r_f) + a^*(r_G - r_f)]^\gamma} + \frac{(1-\pi)(r_B - r_f)}{[Y_0(1+r_f) + a^*(r_B - r_f)]^\gamma} = 0$$

$$\begin{aligned} & \pi(r_G - r_f)[Y_0(1+r_f) + a^*(r_B - r_f)]^\gamma \\ = \;& (1-\pi)(r_f - r_B)[Y_0(1+r_f) + a^*(r_G - r_f)]^\gamma \end{aligned}$$

$$\begin{aligned} & [\pi(r_G - r_f)]^{1/\gamma}[Y_0(1+r_f) + a^*(r_B - r_f)] \\ = \;& [(1-\pi)(r_f - r_B)]^{1/\gamma}[Y_0(1+r_f) + a^*(r_G - r_f)] \end{aligned}$$

# Portfolios, Risk Aversion, and Wealth

$$\begin{aligned}
& [\pi(r_G - r_f)]^{1/\gamma}[Y_0(1+r_f) + a^*(r_B - r_f)] \\
= \;& [(1-\pi)(r_f - r_B)]^{1/\gamma}[Y_0(1+r_f) + a^*(r_G - r_f)]
\end{aligned}$$

$$\begin{aligned}
& Y_0(1+r_f)[\pi(r_G - r_f)]^{1/\gamma} + a^*(r_B - r_f)[\pi(r_G - r_f)]^{1/\gamma} \\
= \;& Y_0(1+r_f)[(1-\pi)(r_f - r_B)]^{1/\gamma} \\
& + a^*(r_G - r_f)[(1-\pi)(r_f - r_B)]^{1/\gamma}
\end{aligned}$$

$$\begin{aligned}
& Y_0(1+r_f)\{[\pi(r_G - r_f)]^{1/\gamma} - [(1-\pi)(r_f - r_B)]^{1/\gamma}]\} \\
= \;& a^*\{(r_G - r_f)[(1-\pi)(r_f - r_B)]^{1/\gamma} + (r_f - r_B)[\pi(r_G - r_f)]^{1/\gamma}\}
\end{aligned}$$

# Portfolios, Risk Aversion, and Wealth

$$\begin{aligned} & Y_0(1+r_f)\{[\pi(r_G - r_f)]^{1/\gamma} - [(1-\pi)(r_f - r_B)]^{1/\gamma}]\} \\ = \; & a^*\{(r_G - r_f)[(1-\pi)(r_f - r_B)]^{1/\gamma} + (r_f - r_B)[\pi(r_G - r_f)]^{1/\gamma}\} \end{aligned}$$

implies

$$\frac{a^*}{Y_0} = \frac{(1+r_f)\{[\pi(r_G - r_f)]^{1/\gamma} - [(1-\pi)(r_f - r_B)]^{1/\gamma}]\}}{(r_G - r_f)[(1-\pi)(r_f - r_B)]^{1/\gamma} + (r_f - r_B)[\pi(r_G - r_f)]^{1/\gamma}}$$

# Portfolios, Risk Aversion, and Wealth

$$\frac{a^*}{Y_0} = \frac{(1+r_f)\{[\pi(r_G - r_f)]^{1/\gamma} - [(1-\pi)(r_f - r_B)]^{1/\gamma}]\}}{(r_G - r_f)[(1-\pi)(r_f - r_B)]^{1/\gamma} + (r_f - r_B)[\pi(r_G - r_f)]^{1/\gamma}}$$

| $\gamma$ | $r_f$ | $r_G$ | $r_B$ | $\pi$ | $E(\tilde{r})$ | $a^*/Y_0$ |
|---|---|---|---|---|---|---|
| 0.5 | 0.05 | 0.40 | −0.20 | 0.50 | 0.10 | 1.20 |
| 1 | 0.05 | 0.40 | −0.20 | 0.50 | 0.10 | 0.60 |
| 2 | 0.05 | 0.40 | −0.20 | 0.50 | 0.10 | 0.30 |
| 3 | 0.05 | 0.40 | −0.20 | 0.50 | 0.10 | 0.20 |
| 5 | 0.05 | 0.40 | −0.20 | 0.50 | 0.10 | 0.12 |
| 10 | 0.05 | 0.40 | −0.20 | 0.50 | 0.10 | 0.06 |

# Portfolios, Risk Aversion, and Wealth

| $\gamma$ | $r_f$ | $r_G$ | $r_B$ | $\pi$ | $E(\tilde{r})$ | $a^*/Y_0$ |
|---|---|---|---|---|---|---|
| 0.5 | 0.05 | 0.40 | $-0.20$ | 0.50 | 0.10 | 1.20 |
| 1 | 0.05 | 0.40 | $-0.20$ | 0.50 | 0.10 | 0.60 |
| 2 | 0.05 | 0.40 | $-0.20$ | 0.50 | 0.10 | 0.30 |
| 3 | 0.05 | 0.40 | $-0.20$ | 0.50 | 0.10 | 0.20 |
| 5 | 0.05 | 0.40 | $-0.20$ | 0.50 | 0.10 | 0.12 |
| 10 | 0.05 | 0.40 | $-0.20$ | 0.50 | 0.10 | 0.06 |

Consistent with Arrow's theorem, higher coefficients of relative risk aversion are associated with smaller values of $a^*$.

# Portfolios, Risk Aversion, and Wealth

$$\frac{a^*}{Y_0} = \frac{(1 + r_f)\{[\pi(r_G - r_f)]^{1/\gamma} - [(1-\pi)(r_f - r_B)]^{1/\gamma}]\}}{(r_G - r_f)[(1-\pi)(r_f - r_B)]^{1/\gamma} + (r_f - r_B)[\pi(r_G - r_f)]^{1/\gamma}}$$

Also consistent with Arrow's results, we see here that with constant relative risk aversion, $a^*$ rises proportionally with wealth.