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FINANCIAL ECONOMICS

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Consumer Optimization: The Risk Dimension

We've already seen how contingent claims can be used to replicate the stock and the bond.

Now let's see how the stock and the bond can be used to replicate the contingent claims.

Consumer Optimization: The Risk Dimension

Consider buying s shares of stock and b bonds, in order to replicate the contingent claim for the good state.

In the good state, the payoffs should be

$$sd^G + b = 1$$

and in the bad state, the payoffs should be

$$sd^B + b = 0$$

since the contingent claim pays off one in the good state and zero in the bad state.

Consumer Optimization: The Risk Dimension

To replicate the contingent claim for the good state:

$$sd^G + b = 1$$

$$sd^B + b = 0 \Rightarrow b = -sd^B$$

Substitute the second equation into the first to solve for

$$s = \frac{1}{d^G - d^B} \text{ and } b = \frac{-d^B}{d^G - d^B}$$

Since s and b are of opposite sign, this requires going "long" one asset and "short" the other.

Consumer Optimization: The Risk Dimension

To replicate the contingent claim for the good state:

$$s = \frac{1}{d^G - d^B} \text{ and } b = \frac{-d^B}{d^G - d^B}$$

If we know the prices q^{stock} and q^{bond} of the stock and bond, we can infer that in the absence of arbitrage, the claim for the good state would have price

$$q^G = q^{stock}s + q^{bond}b = \frac{q^{stock} - d^B q^{bond}}{d^G - d^B}.$$

Consumer Optimization: The Risk Dimension

Consider buying s shares of stock and b bonds, in order to replicate the contingent claim for the bad state.

In the good state, the payoffs should be

$$sd^G + b = 0$$

and in the bad state, the payoffs should be

$$sd^B + b = 1$$

since the contingent claim pays off one in the bad state and zero in the good state.

Consumer Optimization: The Risk Dimension

To replicate the contingent claim for the bad state:

$$sd^G + b = 0 \Rightarrow b = -sd^G$$

$$sd^B + b = 1$$

Substitute the first equation into the second to solve for

$$s = \frac{-1}{d^G - d^B} \text{ and } b = \frac{d^G}{d^G - d^B}$$

Once again, this requires going long one asset and short the other.

Consumer Optimization: The Risk Dimension

To replicate the contingent claim for the bad state:

$$s = \frac{-1}{d^G - d^B} \text{ and } b = \frac{d^G}{d^G - d^B}$$

Once again, if we know the prices q^{stock} and q^{bond} of the stock and bond, we can infer that in the absence of arbitrage, the claim for the bad state would have price

$$q^B = q^{stock}s + q^{bond}b = \frac{d^G q^{bond} - q^{stock}}{d^G - d^B}.$$

Consumer Optimization: The Risk Dimension

What makes it possible to go back and forth between traded assets, like stocks and bonds, and contingent claims is that there are the same number of traded assets as there are possible states of the world next year.

More generally, asset markets are complete if there are as many assets (with linearly independent payoffs) as there are states next year.

Consumer Optimization: The Risk Dimension

If asset markets are complete, then we can use the prices of traded assets to infer the prices of contingent claims.

Then we can use the contingent claims prices to infer the price of any newly-introduced asset.

Black-Scholes Option Pricing

A call option is a contract that gives the buyer the right, but not the obligation, to purchase a share of stock at the strike price K at $t = 1$.

At $t = 1$, the call is said to be in the money if the actual share price is above the strike price and out of the money if the actual share price is below the strike price.

At $t = 1$, the option will have value only if it is in the money. But at $t = 0$, the option will have value even if there is only a probability of it being in the money at $t = 1$.

Black-Scholes Option Pricing

Fischer Black (US, 1938-1995) and Myron Scholes (Canada/US, b.1941, Nobel Prize 1997) were the first to derive a formula for the price of an option.

Robert Merton (US, b.1944, Nobel Prize 1997) arrived at the same formula in a simpler way, by showing how options prices could be inferred from assumptions about and observations on the underlying stock price.

Black-Scholes Option Pricing

The arguments used by Merton were not exactly those from Arrow-Debreu no-arbitrage theory that would use the price of the stock and bond to infer contingent claims prices, then use contingent claims prices to compute the price of the option.

But his analysis followed along similar lines, and today it is recognized that one could use the Arrow-Debreu approach to obtain the same results.

Black-Scholes Option Pricing

Their papers were both published in 1973.

Fischer Black and Myron Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* Vol.81 (May-June 1973): pp.637-654.

Robert Merton, "Theory of Rational Option Pricing," *The Bell Journal of Economics and Management Science* Vol.4 (Spring 1973): pp.141-183.