# ECON 337901
# FINANCIAL ECONOMICS

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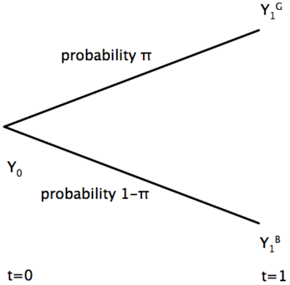
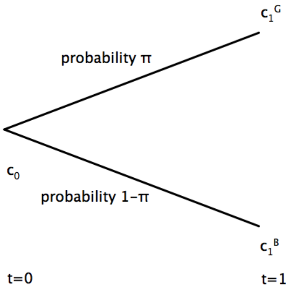
# Consumer Optimization: The Risk Dimension




An event tree highlights randomness in income as the source of risk.

# Consumer Optimization: The Risk Dimension




Under uncertainty, the consumer chooses consumption today and consumption in both states next year.

# Consumer Optimization: The Risk Dimension

The consumer's preferences are described by the expected utility function

$$u(c_0) + \beta\pi u(c_1^G) + \beta(1-\pi)u(c_1^B),$$

so that the terms involving next year's consumption are weighted by the probability that each state will occur as well as by the discount factor $\beta$ and the concavity of $u$ captures the consumer's risk aversion.

# Consumer Optimization: The Risk Dimension

But how do consumers implement state-contingent consumption plans?

Arrow and Debreu answered this question by imagining that consumers trade in contingent claims.

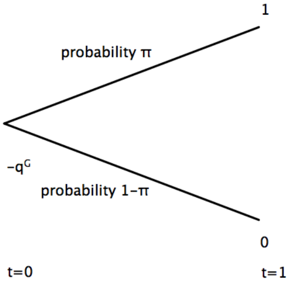
# Consumer Optimization: The Risk Dimension

To implement these state-contingent consumption plans, Arrow and Debreu imagined that the consumer would trade contingent claims for both future states.

A contingent claim for the good state costs $q^G$ today, and delivers one unit of consumption next year in the good state and zero units of consumption next year in the bad state.

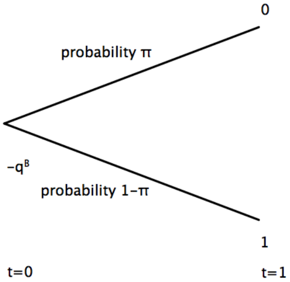
A contingent claim for the bad state costs $q^B$ today, and delivers one unit of consumption next year in the bad state and zero units of consumption next year in the good state.

# Consumer Optimization: The Risk Dimension



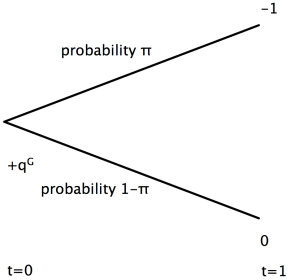

Payoffs for the contingent claim for the good state (a long position).

# Consumer Optimization: The Risk Dimension



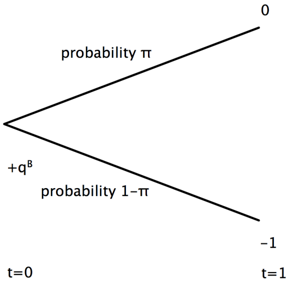

Payoffs for the contingent claim for the bad state (a long position).

# Consumer Optimization: The Risk Dimension




Payoffs for a short position in the contingent claim for the good state.

# Consumer Optimization: The Risk Dimension




Payoffs for a short position in the contingent claim for the bad state.

# Consumer Optimization: The Risk Dimension

Today, the consumer divides his or her income up into an amount to be consumed and amounts used to purchase the two contingent claims:

$$Y_0 \geq c_0 + q^G s^G + q^B s^B,$$

where $s^G$ and $s^B$ denote the number of each contingent claim purchased or sold short.

If either $s^G$ or $s^B$ is negative, the consumer is taking a short position in that claim.

# Consumer Optimization: The Risk Dimension

Next year, the consumer simply spends his or her income, including payoffs on contingent claims:

$$Y_1^G + s^G \geq c_1^G$$

in the good state and

$$Y_1^B + s^B \geq c_1^B$$

in the bad state.

# Consumer Optimization: The Risk Dimension

$$Y_0 \geq c_0 + q^G s^G + q^B s^B$$
$$Y_1^G + s^G \geq c_1^G$$
$$Y_1^B + s^B \geq c_1^B$$

Multiply both sides of the second equation by $q^G$ and both sides of the third equation by $q^B$, Then add them all up to get the lifetime budget constraint

$$Y_0 + q^G Y_1^G + q^B Y_1^B \geq c_0 + q^G c_1^G + q^B c_1^B.$$

# Consumer Optimization: The Risk Dimension

The problem is to choose $c_0$, $c_1^G$, and $c_1^B$ to maximize expected utility

$$u(c_0) + \beta\pi u(c_1^G) + \beta(1-\pi)u(c_1^B),$$

subject to the budget constraint

$$Y_0 + q^G Y_1^G + q^B Y_1^B \geq c_0 + q^G c_1^G + q^B c_1^B.$$

This was Arrow and Debreu's key insight: that finance is like grocery shopping. Mathematically, making decisions over time and under uncertainty is no different from choosing apples, bananas, and pears!

# Consumer Optimization: The Risk Dimension

The Lagrangian is

$$
\begin{aligned}
L &= u(c_0) + \beta\pi u(c_1^G) + \beta(1-\pi)u(c_1^B) \\
&\quad + \lambda\left(Y_0 + q^G Y_1^G + q^B Y_1^B - c_0 - q^G c_1^G - q^B c_1^B\right),
\end{aligned}
$$

and the first-order conditions are

$$u'(c_0^*) - \lambda^* = 0$$

$$\beta\pi u'(c_1^{G*}) - \lambda^* q^G = 0$$

$$\beta(1-\pi)u'(c_1^{B*}) - \lambda^* q^B = 0$$

# Consumer Optimization: The Risk Dimension

The first-order conditions

$$u'(c_0^*) - \lambda^* = 0$$

$$\beta\pi u'(c_1^{G*}) - \lambda^* q^G = 0$$

$$\beta(1-\pi)u'(c_1^{B*}) - \lambda^* q^B = 0$$

imply that marginal rates of substitution equal relative prices:

$$\frac{u'(c_0^*)}{\beta\pi u'(c_1^{G*})} = \frac{1}{q^G} \text{ and } \frac{u'(c_0^*)}{\beta(1-\pi)u'(c_1^{B*})} = \frac{1}{q^B}$$

$$\text{and } \frac{\pi u'(c_1^{G*})}{(1-\pi)u'(c_1^{B*})} = \frac{q^G}{q^B}.$$

# Consumer Optimization: The Risk Dimension

Do we really observe consumers trading in contingent claims?

Yes, if we think of financial assets as "bundles" of contingent claims.

This insight is also Arrow and Debreu's.