ECON 337901
FINANCIAL ECONOMICS

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Consumer Optimization: Algebraic Analysis

The consumer chooses c_0, c_1, and c_2 to maximize the utility function

$$u(c_0) + \alpha u(c_1) + \beta u(c_2),$$

subject to the budget constraint

$$Y \geq p_0 c_0 + p_1 c_1 + p_2 c_2.$$

The Lagrangian for this problem is

$$L = u(c_0) + \alpha u(c_1) + \beta u(c_2) + \lambda(Y - p_0 c_0 - p_1 c_1 - p_2 c_2).$$

Consumer Optimization: Algebraic Analysis

$$L = u(c_0) + \alpha u(c_1) + \beta u(c_2) + \lambda(Y - p_0 c_0 - p_1 c_1 - p_2 c_2).$$

First-order conditions:

$$u'(c_0^*) - \lambda^* p_0 = 0$$
$$\alpha u'(c_1^*) - \lambda^* p_1 = 0$$
$$\beta u'(c_2^*) - \lambda^* p_2 = 0$$

Consumer Optimization: Algebraic Analysis

The first-order conditions

$$u'(c_0^*) - \lambda^* p_0 = 0$$

$$\alpha u'(c_1^*) - \lambda^* p_1 = 0$$

$$\beta u'(c_2^*) - \lambda^* p_2 = 0$$

imply

$$\frac{u'(c_0^*)}{\alpha u'(c_1^*)} = \frac{p_0}{p_1} \text{ and } \frac{u'(c_0^*)}{\beta u'(c_2^*)} = \frac{p_0}{p_2} \text{ and } \frac{\alpha u'(c_1^*)}{\beta u'(c_2^*)} = \frac{p_1}{p_2}.$$

The marginal rate of substitution equals the relative prices.

Consumer Optimization: The Time Dimension

Irving Fisher (US, 1867-1947) was the first to recognize that the basic theory of consumer decision-making could be used to understand how to optimally allocate spending intertemporally, that is, over time, as well as how to optimally allocate spending across different goods in a static, or point-in-time, analysis.

Consumer Optimization: The Time Dimension

Following Fisher, return to the case of two goods, but reinterpret:

$$c_0 = \text{consumption today}$$

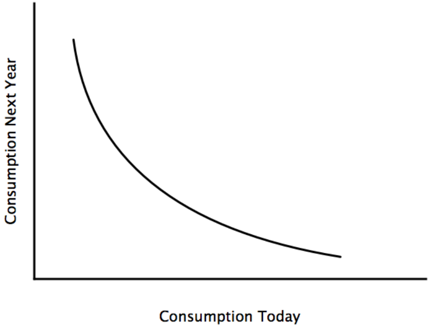
$$c_1 = \text{consumption next year}$$

Suppose that the consumer's utility function is

$$u(c_0) + \beta u(c_1),$$

where β now has a more specific interpretation, as the discount factor, a measure of patience.

Consumer Optimization: The Time Dimension



A concave utility function implies that indifference curves are convex, so that the consumer has a preference for a smoothness in consumption.

Consumer Optimization: The Time Dimension

Next, let

$$\begin{aligned} Y_0 &= \text{income today} \\ Y_1 &= \text{income next year} \\ s &= \text{amount saved (or borrowed if negative) today} \\ r &= \text{interest rate} \end{aligned}$$

Consumer Optimization: The Time Dimension

Today, the consumer divides his or her income up into an amount to be consumed and an amount to be saved:

$$Y_0 \geq c_0 + s.$$

Next year, the consumer simply spends his or her income, including interest earnings if s is positive or net of interest expenses if s is negative:

$$Y_1 + (1 + r)s \geq c_1.$$

Consumer Optimization: The Time Dimension

Divide both sides of next year's budget constraint by $1 + r$ to get

$$\frac{Y_1}{1+r} + s \geq \frac{c_1}{1+r}.$$

Now combine this inequality with this year's budget constraint

$$Y_0 \geq c_0 + s.$$

to get

$$Y_0 + \frac{Y_1}{1+r} \geq c_0 + \frac{c_1}{1+r}.$$

Consumer Optimization: The Time Dimension

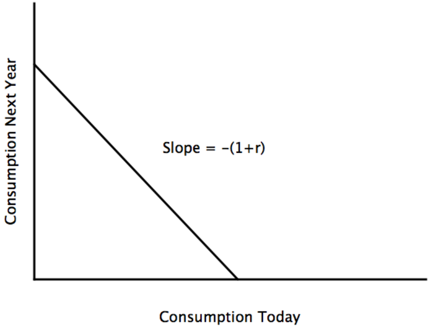
The "lifetime" budget constraint

$$Y_0 + \frac{Y_1}{1+r} \geq c_0 + \frac{c_1}{1+r}$$

says that the present value of income must be sufficient to cover the present value of consumption over the two periods. It also shows that the "price" of consumption today relative to the "price" of consumption next year is related to the interest rate via

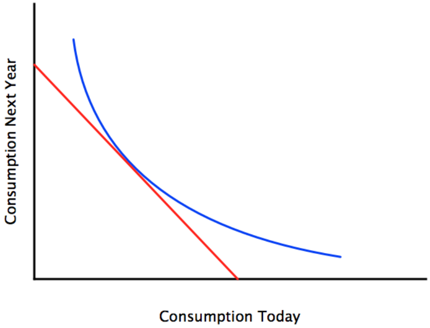
$$\frac{p_0}{p_1} = 1 + r.$$

Consumer Optimization: The Time Dimension



The slope of the intertemporal budget constraint is $-(1 + r)$.

Consumer Optimization: The Time Dimension



At the optimum, the intertemporal marginal rate of substitution equals the slope of the intertemporal budget constraint.

Consumer Optimization: The Time Dimension

We now know the answer ahead of time: if we take an algebraic approach to solve the consumer's problem, we will find that the IMRS equals the slope of the intertemporal budget constraint:

$$\frac{u'(c_0)}{\beta u'(c_1)} = 1 + r.$$

But let's use calculus to derive the same result.

Consumer Optimization: The Time Dimension

The problem is to choose c_0 and c_1 to maximize utility

$$u(c_0) + \beta u(c_1)$$

subject to the budget constraint

$$Y_0 + \frac{Y_1}{1+r} \geq c_0 + \frac{c_1}{1+r}.$$

The Lagrangian is

$$L = u(c_0) + \beta u(c_1) + \lambda \left(Y_0 + \frac{Y_1}{1+r} - c_0 - \frac{c_1}{1+r} \right).$$

Consumer Optimization: The Time Dimension

$$L = u(c_0) + \beta u(c_1) + \lambda \left(Y_0 + \frac{Y_1}{1+r} - c_0 - \frac{c_1}{1+r} \right).$$

The first-order conditions

$$u'(c_0^*) - \lambda^* = 0$$

$$\beta u'(c_1^*) - \lambda^* \left(\frac{1}{1+r} \right) = 0.$$

lead directly to the graphical result

$$\frac{u'(c_0^*)}{\beta u'(c_1^*)} = 1 + r.$$