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FINANCIAL ECONOMICS

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Deriving the CAPM

Modern portfolio theory with a risk-free asset implies that all individual investors demand risky assets only to the extent that those assets comprise the tangency portfolio.

The Capital Asset Pricing Model then observes that, in equilibrium, the tangency portfolio must coincide with the market portfolio.

More precisely, asset prices must adjust to equate the tangency portfolio with the market portfolio, just as in basic microeconomics, prices must adjust to equate the demand and supply of any good or service.

Deriving the CAPM

Likewise, the price of any individual stock must adjust so that investors hold that stock in the same proportion to its importance in the market portfolio.

Otherwise, buying or selling would put upward or downward pressure on the stock price until equilibrium is restored.

What, precisely, does this imply for the prices of or expected returns on individual stocks? This is what Sharpe, Lintner, and Mossin figured out in the mid-1960s.

CAPM Implications

The CAPM implies that for all individual stocks and portfolios of stocks

$$E(\tilde{r}_j) = r_f + \beta_j[E(\tilde{r}_M) - r_f]$$

Where each stock's CAPM beta

$$\beta_j = \frac{\sigma_{jM}}{\sigma_M^2}$$

depends on its return's covariance σ_{jM} with the market, not on its variance σ_j^2.

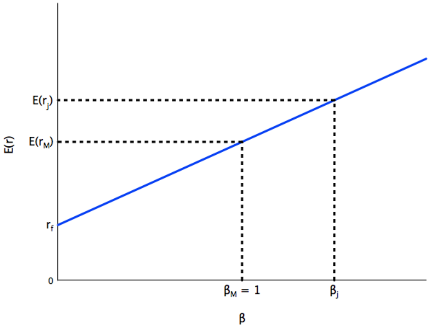
CAPM Implications

$$E(\tilde{r}_j) = r_f + \beta_j[E(\tilde{r}_M) - r_f]$$

This equation summarizes a very strong restriction: Given r_f and $E(\tilde{r}_M)$, each stock's expected return (and hence its price today) depends on β_j and only on β_j.

And it also implies that if we rank individual stocks or portfolios of stocks according to their betas, their expected returns should all lie along a single security market line with slope $E(\tilde{r}_M) - r_f$.

CAPM Implications



According to the CAPM, all assets and portfolios of assets lie along a single security market line. Those with higher betas have higher expected returns.

CAPM Implications

Eugene Fama (Nobel Prize 2013) has tested these implications of the CAPM extensively, with mixed results.

Early work shows that when stocks are ranked according to their CAPM betas, their expected returns tend to line up on a SML as predicted by the theory.

Eugene Fama and James MacBeth, "Risk, Return, and Equilibrium," *Journal of Political Economy* Vol.81 (May-June 1973), pp.607-636.

CAPM Implications

As a practical matter, professional funds managers have generally struggled to deliver returns on actively managed portfolios that have higher Sharpe ratios than the market.

Eugene Fama. *Foundations of Finance*, 1976.

`https://faculty.chicagobooth.edu/eugene.fama/research/index.htm`

Burton Malkiel, *A Random Walk Down Wall Street*.

CAPM Implications

But more recent evidence contradicts the CAPM's implication that beta is the only determinant of differences in expected returns across different stocks.

Eugene Fama and Kenneth French, "Common Risk Factors in the Returns on Stocks and Bonds," *Journal of Financial Economics* Vol.33 (February 1993): pp.3-56.

This paper shows that equity shares in small firms and in firms with high book (accounting) to market value have expected returns that differ strongly from what is predicted by the CAPM alone.

CAPM Implications

The "Fama-French three-factor model" explains differences in expected returns across stocks using (1) the CAPM beta, (2) firm size, and (3) book-to-market value.

Intriguingly, the most successful investors like Peter Lynch and Warren Buffett, have advocated strategies involving small firms or "value stocks."

Quite a bit of recent research has been directed towards understanding the source of these "anomalies."

Interpreting the CAPM

$$E(\tilde{r}_j) = r_f + \beta_j[E(\tilde{r}_M) - r_f]$$

There are several complementary ways of interpreting this result:

1. Low beta as an amenity.
2. Aggregate versus idiosyncratic risk.
3. CAPM and regression betas.

Low Beta as an Amenity

The first interpretation goes directly back to the MPT: a stock with low and especially negative σ_{jM} will be most useful for diversification.

But then all investors will want to hold that stock. In equilibrium, therefore, the stock's price will be high and, given future cash flows, its expected return will be low.

Therefore, stocks with low or negative betas will have low expected returns. Investors hold these stocks, despite their low expected returns, because of they are useful for diversification.

Low Beta as an Amenity

Conversely, a stock with high, positive σ_{jM} will not be very useful for diversification.

In equilibrium, therefore, the stock will sell for a low price.

Therefore, stocks with high betas will have high expected returns. The high expected return is needed to compensate investors, because the stock is not very useful for diversification.

Aggregate versus Idiosyncratic Risk

The second interpretation also reiterates the MPT's lessons about diversification, but goes further by distinguishing between two types of risk.

Idiosyncratic, or firm-specific, risk can be "diversified away" and therefore does not get "priced."

But aggregate risk, that is correlated with the market, cannot be diversified away. Investors must therefore be compensated for bearing this risk.

Aggregate versus Idiosyncratic Risk

This second interpretation combines CAPM equation

$$E(\tilde{r}_j) = r_f + \left(\frac{\sigma_{jM}}{\sigma_M^2}\right)[E(\tilde{r}_M) - r_f]$$

with the definition of correlation, which implies $\sigma_{jM} = \sigma_j \sigma_M \rho_{jM}$, to re-express the CAPM relationship as

$$\begin{aligned} E(\tilde{r}_j) &= r_f + \left(\frac{\sigma_j \sigma_M \rho_{jM}}{\sigma_M^2}\right)[E(\tilde{r}_M) - r_f] \\ &= r_f + \left[\frac{E(\tilde{r}_M) - r_f}{\sigma_M}\right]\sigma_j \rho_{jM} \end{aligned}$$

Aggregate versus Idiosyncratic Risk

$$E(\tilde{r}_j) = r_f + \left[\frac{E(\tilde{r}_M) - r_f}{\sigma_M}\right] \sigma_j \rho_{jM}$$

The market's Sharpe ratio is the "equilibrium price of risk."

But since the correlation ρ_{jM} lies between -1 and $+1$, it represents the "fraction" of "total risk" σ_j that is "priced."

Aggregate versus Idiosyncratic Risk

$$E(\tilde{r}_j) = r_f + \left[\frac{E(\tilde{r}_M) - r_f}{\sigma_M}\right] \sigma_j \rho_{jM}$$

The idiosyncratic risk in asset j, that is, the portion that is uncorrelated with the market return, can be diversified away by holding many individual stocks and relying on the law of large numbers, that is, by not putting all your eggs in one basket.

Since this risk can be freely shed through diversification, it is not "priced."

Aggregate versus Idiosyncratic Risk

$$E(\tilde{r}_j) = r_f + \left[\frac{E(\tilde{r}_M) - r_f}{\sigma_M}\right] \sigma_j \rho_{jM}$$

But aggregate risk, because it is correlated with market return, cannot be diversified away. Investors must be compensated for tolerating this risk.

Aggregate versus Idiosyncratic Risk

$$E(\tilde{r}_j) = r_f + \left[\frac{E(\tilde{r}_M) - r_f}{\sigma_M}\right] \sigma_j \rho_{jM}$$

Thus, according to the CAPM:

1. Only assets with random returns that are positively correlated with the market return earn expected returns above the risk free rate. They must, in order to induce investors to take on more aggregate risk.

Aggregate versus Idiosyncratic Risk

$$E(\tilde{r}_j) = r_f + \left[\frac{E(\tilde{r}_M) - r_f}{\sigma_M}\right] \sigma_j \rho_{jM}$$

Note that in PS13, Q4, high beta stocks tend to be in industries that are highly cyclical. Investors must be compensated by higher expected returns for holding them. Low beta stocks tend to be less affected by macroeconomic conditions. Investors are willing to hold them, despite lower expected returns.

Aggregate versus Idiosyncratic Risk

$$E(\tilde{r}_j) = r_f + \left[\frac{E(\tilde{r}_M) - r_f}{\sigma_M}\right] \sigma_j \rho_{jM}$$

Thus, according to the CAPM:

2. Assets with returns that are uncorrelated with the market return have expected returns equal to the risk free rate, since their risk can be completely diversified away.

Aggregate versus Idiosyncratic Risk

$$E(\tilde{r}_j) = r_f + \left[\frac{E(\tilde{r}_M) - r_f}{\sigma_M}\right] \sigma_j \rho_{jM}$$

Thus, according to the CAPM:

3. Assets with negative betas – that is, with random returns that are negatively correlated with the market return – have expected returns below the risk free rate! For these assets, $E(\tilde{r}_j) - r_f < 0$ is like an "insurance premium" that investors will pay in order to insulate themselves from aggregate risk. Gold may be an example.

CAPM and Regression Betas

The third interpretation is based on a statistical regression of the random return $\tilde{r}_j$ on asset j on a constant and the market return $\tilde{r}_M$:

$$\tilde{r}_j = \alpha + \beta_j \tilde{r}_M + \varepsilon_j$$

This regression breaks the variance of $\tilde{r}_j$ down into two "orthogonal" (uncorrelated) components:

1. The component $\beta_j \tilde{r}_M$ that is systematically related to variation in the market return.
2. The component ε_j that is not.

Do you remember the formula for β_j, the slope coefficient in a linear regression?

CAPM and Regression Betas

Consider a statistical regression of the random return $\tilde{r}_j$ on asset j on a constant and the market return $\tilde{r}_M$:

$$\tilde{r}_j = \alpha + \beta_j \tilde{r}_M + \varepsilon_j$$

Do you remember the formula for β_j, the slope coefficient in a linear regression? It is

$$\beta_j = \frac{\sigma_{jM}}{\sigma_M^2}$$

the same "beta" as in the CAPM!

CAPM and Regression Betas

Consider a statistical regression:

$$\tilde{r}_j = \alpha + \beta_j \tilde{r}_M + \varepsilon_j \text{ with } \beta_j = \sigma_{jM}/\sigma_M^2$$

the same "beta" as in the CAPM!

But this is not an accident: to the contrary, it restates the conclusion that, according to the CAPM, risk in an individual asset is priced – and thereby reflected in a higher expected return – only to the extent that it is correlated with the market return.

Valuing Risky Cash Flows

Previously, we saw that a risk-free asset that pays off C_{t+1}, for sure, one year from now has price

$$P_t^A = \frac{C_{t+1}}{1 + r_f}$$

today, where r_f is the interest rate on a risk-free one-year discount bond.

Valuing Risky Cash Flows

To extend this asset-pricing principle to apply to a risky asset that makes a risky (random) payment $\tilde{C}_{t+1}$, we want to discount the expected cashflow by a larger amount to "penalize" the asset for its riskiness:

$$P_t^C = \frac{E(\tilde{C}_{t+1})}{1 + r_f + \psi_c}$$

Now, as promised, we can use the CAPM to determine the appropriate risk premium ψ_c.

Valuing Risky Cash Flows

Because the CAPM's implications are stated in terms of expected returns, we need to start by computing the risky return $\tilde{r}_c$ and the expected return $E(\tilde{r}_c)$ on the asset that costs P_t^C today and makes the random payment $\tilde{C}_{t+1}$ next year:

$$\tilde{r}_c = \frac{\tilde{C}_{t+1} - P_t^C}{P_t^C} = \frac{\tilde{C}_{t+1}}{P_t^C} - 1.$$

$$E(\tilde{r}_c) = E\left(\frac{\tilde{C}_{t+1}}{P_t^C} - 1\right) = \frac{E(\tilde{C}_{t+1})}{P_t^C} - 1$$

Valuing Risky Cash Flows

The CAPM implies that the expected return $E(\tilde{r}_c)$ must satisfy

$$E(\tilde{r}_c) = r_f + \beta_c[E(\tilde{r}_M) - r_f]$$

where the asset's beta depends on the covariance of its return with the market return:

$$\beta_c = \frac{\sigma_{CM}}{\sigma_M^2}$$

This is what takes skill: with an existing asset, one can use data on the past correlation between its return and the market return to estimate beta. With a totally new asset, a combination of experience, creativity, and hard work is often needed to choose the right value for β_c.

Valuing Risky Cash Flows

But once a value for β_c is determined and $E(\tilde{r}_c)$ has been estimated, we can use the CAPM

$$E(\tilde{r}_c) = r_f + \beta_c[E(\tilde{r}_M) - r_f]$$

together with the equation linking the expected return to the asset price

$$E(\tilde{r}_c) = \frac{E(\tilde{C}_{t+1})}{P_t^C} - 1$$

to write

$$1 + r_f + \beta_c[E(\tilde{r}_M) - r_f] = \frac{E(\tilde{C}_{t+1})}{P_t^C}$$

Valuing Risky Cash Flows

Finally, rewrite

$$1 + r_f + \beta_c[E(\tilde{r}_M) - r_f] = \frac{E(\tilde{C}_{t+1})}{P_t^C}$$

as

$$P_t^C = \frac{E(\tilde{C}_{t+1})}{1 + r_f + \beta_c[E(\tilde{r}_M) - r_f]} = \frac{E(\tilde{C}_{t+1})}{1 + r_f + \psi_c}$$

to see that CAPM implies a risk premium of

$$\psi_c = \beta_c[E(\tilde{r}_M) - r_f]$$

which depends on $\beta_c = \sigma_{CM}/\sigma_M^2$ and can even be negative if the new asset has a return that covaries negatively with the market.

Strengths and Shortcomings of the CAPM

Econometric testing of the CAPM's implications has produced mixed results.

Expected returns do depend on beta (Fama and MacBeth 1973).

But firm size and book-to-market values seem to matter, too, even after "controlling for beta" (Fama and French 1993).

Strengths and Shortcomings of the CAPM

Despite some empirical shortcomings, however, the CAPM quite usefully deepens our understanding of the gains from diversification.

Related, the CAPM alerts us to the important distinction between idiosyncratic risk, which can be diversified away, and aggregate risk, which cannot.

Like expected utility in microeconomic theory, the CAPM is still widely used in academic research and in practice.