ECON 337901
FINANCIAL ECONOMICS

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April 28, 2020

7 The Capital Asset Pricing Model

- A MPT and the CAPM
- B Deriving the CAPM
- C Valuing Risky Cash Flows
- D Strengths and Shortcomings of the CAPM

Asset Prices and Expected Returns

The Capital Asset Pricing Model (CAPM) is usually described in terms of its implications for expected returns on individual stocks.

But the CAPM is a Capital Asset Pricing Model.

This is because, given future cash flows, an asset's price and its expected return are closely (though inversely) related.

Asset Prices and Expected Returns

Consider a stock that sells for price P_0^s today (time 0) and makes a single risky (random) payment $\tilde{C}_1$ next year (time 1).

As we've discussed before, next year's cash flow $\tilde{C}_1$ can consist of a dividend payment $(\tilde{D}_1)$, a capital gain $(\tilde{P}_1^s)$, or any combination of the two.

And, as we've discussed before, adding more cash flows $\tilde{C}_2, \tilde{C}_3, \ldots \tilde{C}_T$ received in additional future periods is conceptually straightforward.

Asset Prices and Expected Returns

Consider a stock that sells for price P_0^s today (time 0) and makes a single risky (random) payment $\tilde{C}_1$ next year (time 1).

The stock's random return $\tilde{r}^s$ is

$$\tilde{r}^s = \frac{\tilde{C}_1 - P_0^s}{P_0^s} = \frac{\tilde{C}_1}{P_0^s} - 1$$

And the stock's expected return μ^s is

$$\mu^s = E(\tilde{r}^s) = E\left(\frac{\tilde{C}_1}{P_0^s} - 1\right) = \frac{E(\tilde{C}_1)}{P_0^s} - 1$$

Asset Prices and Expected Returns

Given an estimate $E(\tilde{C}_1)$ of the future cash flow, knowing the asset price P_0^s lets us compute the expected return

$$E(\tilde{r}^s) = \frac{E(\tilde{C}_1)}{P_0^s} - 1$$

Or knowing the expected return $E(\tilde{r}^s)$ lets us compute the asset price:

$$1 + E(\tilde{r}^s) = \frac{E(\tilde{C}_1)}{P_0^s}$$

$$P_0^s = \frac{E(\tilde{C}_1)}{1 + E(\tilde{r}^s)}$$

Asset Prices and Expected Returns

$$E(\tilde{r}^s) = \frac{E(\tilde{C}_1)}{P_0^s} - 1$$

$$P_0^s = \frac{E(\tilde{C}_1)}{1 + E(\tilde{r}^s)}$$

Either way, the asset price and its expected return are inversely related.

Given the expected future cash flow, the only way the expected return can go up is if the asset price falls.

Of course, obtaining the estimate $E(\tilde{C}_1)$ will require some work!

MPT and the CAPM

The Capital Asset Pricing Model builds directly on Modern Portfolio Theory.

It was developed in the mid-1960s by William Sharpe (US, b.1934, Nobel Prize 1990), John Lintner (US, 1916-1983), and Jan Mossin (Norway, 1936-1987).

MPT and the CAPM

William Sharpe, "Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk," *Journal of Finance* Vol.19 (September 1964): pp.425-442.

John Lintner, "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," *Review of Economics and Statistics* Vol.47 (February 1965): pp.13-37.

Jan Mossin, "Equilibrium in a Capital Asset Market," *Econometrica* Vol.34 (October 1966): pp.768-783.

MPT and the CAPM

But whereas Modern Portfolio Theory is a theory describing the demand for financial assets, the Capital Asset Pricing Model is a theory describing equilibrium in financial markets.

By making an additional assumption – namely, that supply equals demand in financial markets – the CAPM yields additional implications about the pricing of financial assets and risky cash flows.

MPT and the CAPM

Like MPT, the CAPM assumes that investors have mean-variance utility and hence that either investors have quadratic Bernoulli utility functions or that the random returns on risky assets are normally distributed.

Thus, some of the same caveats that apply to MPT also apply to the CAPM.

That's why people say, "you can't use the CAPM to price options."

MPT and the CAPM

The traditional CAPM also assumes that there is a risk free asset as well as a potentially large collection of risky assets.

Under these circumstances, as we've seen, all investors will hold some combination of the riskless asset and the tangency portfolio: the efficient portfolio of risky assets with the highest Sharpe ratio.

MPT and the CAPM

But the CAPM goes further than the MPT by imposing an equilibrium condition.

Because there is no demand for risky financial assets except to the extent that they comprise the tangency portfolio, and because, in equilibrium, the supply of financial assets must equal demand, the market portfolio consisting of all existing financial assets must coincide with the tangency portfolio.

In equilibrium, that is, "everyone" must "own the market."

MPT and the CAPM

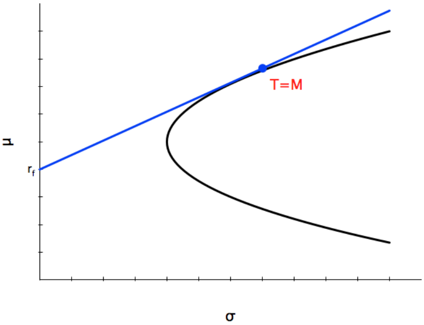
In equilibrium, that is, "everyone" must "own the market."

But why? What happens if not enough people want to "own the market."

Asset prices must adjust so that "everyone owns the market."

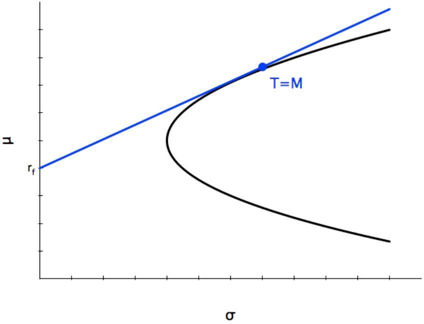
This logic turns the MPT – a theory of asset demand – into the CAPM – an asset pricing model.

Deriving the CAPM



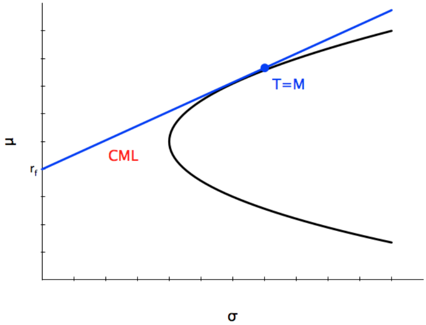
In the CAPM, equilibrium in financial markets requires the demand for risky assets – the tangency portfolio – to coincide with the supply of financial assets – the market portfolio.

Deriving the CAPM



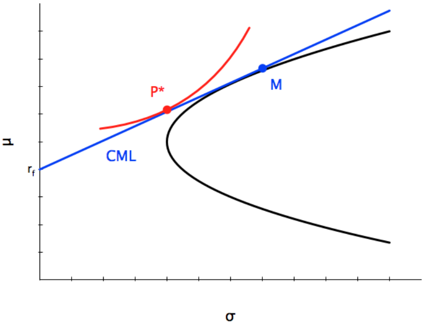
The CAPM's first implications are immediate: the market portfolio lies on the efficient frontier and is the portfolio with the highest Sharpe ratio.

Deriving the CAPM



The line originating at $(0, r_f)$ and running through $(\sigma_M, E(\tilde{r}_M))$ is called the capital market line (CML).

Deriving the CAPM



Hence, it also follows that all individually optimal portfolios are located along the CML and are formed as combinations of the risk free asset and the market portfolio.

Deriving the CAPM

MPT as originally developed by Markowitz implies that portfolio managers should find a portfolio on the efficient frontier.

The separation or two-fund theorem derived later by Tobin implies that portfolio managers should find the portfolio on the efficient frontier with the highest Sharpe ratio.

The CAPM implies that portfolio managers should give up on active management and simply invest in the market portfolio. See also Burton Malkiel's *Random Walk Down Wall Street*.

Deriving the CAPM

Recall that the trade-off between the standard deviation and expected return of any portfolio combining the riskless asset and the tangency portfolio is described by the linear relationship

$$E(\tilde{r}_P) = r_f + \left[\frac{E(\tilde{r}_T) - r_f}{\sigma_T}\right] \sigma_P.$$

Since the CAPM implies that the tangency and market portfolios coincide, the formula for the Capital Market Line is likewise

$$E(\tilde{r}_P) = r_f + \left[\frac{E(\tilde{r}_M) - r_f}{\sigma_M}\right] \sigma_P.$$

Deriving the CAPM

And since all individually optimal portfolios are located along the CML, the equation

$$E(\tilde{r}_P) = r_f + \left[\frac{E(\tilde{r}_M) - r_f}{\sigma_M}\right] \sigma_P.$$

implies that the market portfolio's Sharpe ratio

$$\frac{E(\tilde{r}_M) - r_f}{\sigma_M}$$

measures the equilibrium price of risk: the expected return that each investor gives up when he or she adjusts his or her total portfolio to reduce risk.

Deriving the CAPM

Next, let's consider an arbitrary asset – "asset j" – with random return $\tilde{r}_j$, expected return $E(\tilde{r}_j)$, and standard deviation σ_j.

MPT would take $E(\tilde{r}_j)$ and σ_j as "data" – that is, as given.

The CAPM again goes further and asks: if asset j is to be demanded by investors with mean-variance utility, what restrictions must $E(\tilde{r}_j)$ and σ_j satisfy?

Deriving the CAPM

To answer this question, consider an investor who takes the portion of his or her initial wealth that he or she allocates to risky assets and divides it further: using the fraction w to purchase asset j and the remaining fraction $1 - w$ to buy the market portfolio.

Note that since the market portfolio already includes some of asset j, choosing $w > 0$ really means that the investor "overweights" asset j in his or her own portfolio. Conversely, choosing $w < 0$ means that the investor "underweights" asset j in his or her own portfolio.

Deriving the CAPM

Based on our previous analysis, we know that this investor's portfolio of risky assets now has random return

$$\tilde{r}_P = w\tilde{r}_j + (1-w)\tilde{r}_M,$$

expected return

$$E(\tilde{r}_P) = wE(\tilde{r}_j) + (1-w)E(\tilde{r}_M),$$

and variance

$$\sigma_P^2 = w^2\sigma_j^2 + (1-w)^2\sigma_M^2 + 2w(1-w)\sigma_{jM},$$

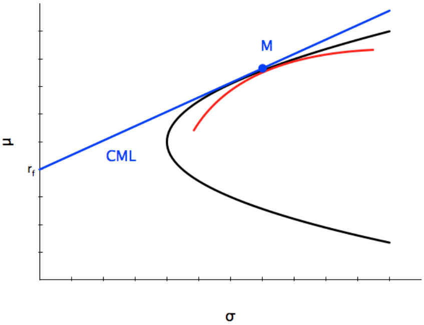
where σ_{jM} is the covariance between $\tilde{r}_j$ and $\tilde{r}_M$.

Deriving the CAPM

$$E(\tilde{r}_P) = wE(\tilde{r}_j) + (1-w)E(\tilde{r}_M),$$
$$\sigma_P^2 = w^2\sigma_j^2 + (1-w)^2\sigma_M^2 + 2w(1-w)\sigma_{jM},$$

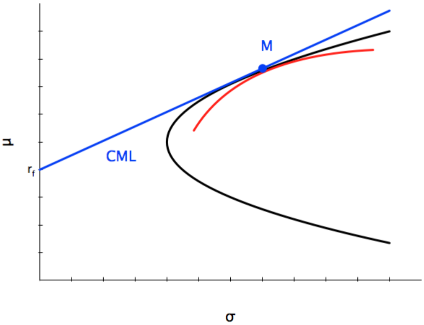
We can use these formulas to trace out how σ_P and $E(\tilde{r}_P)$ vary as w changes.

Deriving the CAPM



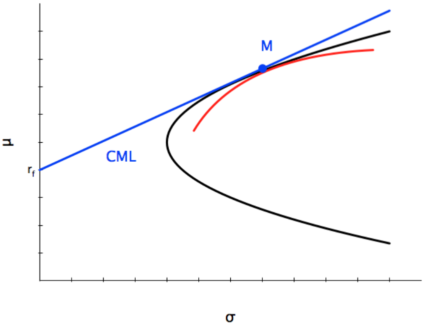
The red curve these traces out how σ_P and $E(\tilde{r}_P)$ vary as w changes, that is, as asset j gets underweighted or overweighted relative to the market portfolio.

Deriving the CAPM



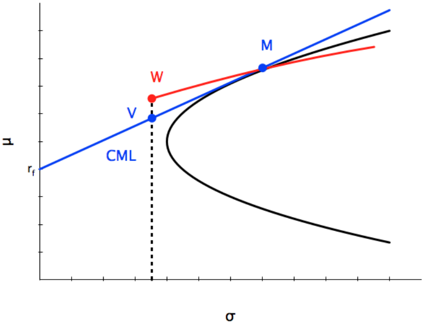
The red curve passes through M, since when $w = 0$ the new portfolio coincides with the market portfolio.

Deriving the CAPM



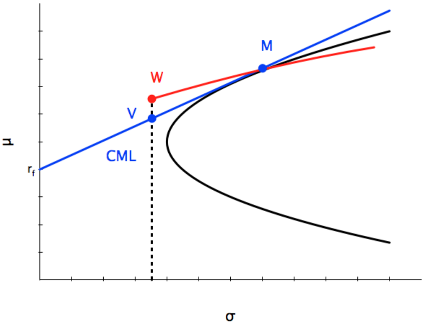
For all other values of w, however, the red curve must lie below the CML.

Deriving the CAPM



Otherwise, a portfolio along the CML would be dominated in mean-variance by the new portfolio. Financial markets would no longer be in equilibrium, since some investors would no longer be willing to hold the market portfolio.

Deriving the CAPM



Suppose that at W, $w > 0$. Then asset j is "undervalued" in the sense that overweighting it will yield a portfolio with a higher expected return.

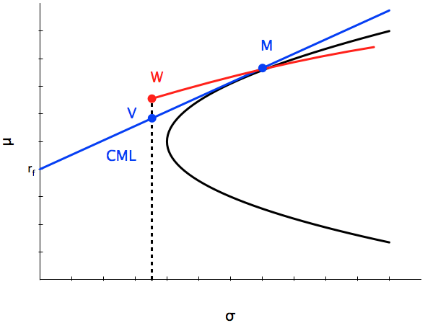
Deriving the CAPM

But as all investors buy this undervalued asset, its price will rise.

Given future cash flows (future price from selling the asset plus any dividends earned), a rise the asset's price will lower its expected return.

Buying pressure will continue until the red curve bends back below the CML.

Deriving the CAPM



Suppose that at W, $w < 0$. Then asset j is "overvalued" in the sense that underweighting it will yield a portfolio with a higher expected return.

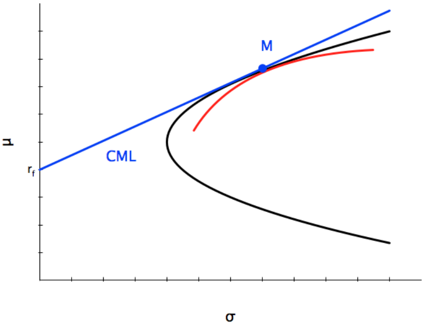
Deriving the CAPM

But as all investors sell this overvalued asset, its price will fall.

Given future cash flows, a fall the asset's price will raise its expected return.

Selling pressure will continue until the red curve bends back below the CML.

Deriving the CAPM



Together, these observations imply that the red curve must be tangent to the CML at M.

Deriving the CAPM

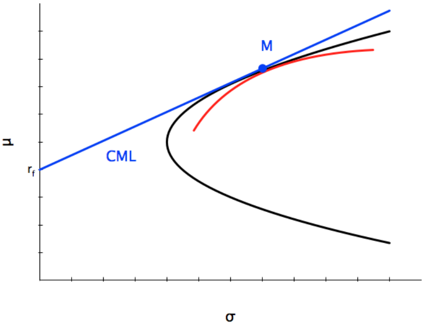
Tangent means equal in slope.

We already know that the slope of the Capital Market Line is

$$\frac{E(\tilde{r}_M) - r_f}{\sigma_M}$$

But what is the slope of the red curve?

Deriving the CAPM



Let $f(\sigma_P)$ be the function defined by $E(\tilde{r}_P) = f(\sigma_P)$ and therefore describing the red curve.

Deriving the CAPM

Next, define the functions $g(w)$ and $h(w)$ by

$$g(w) = wE(\tilde{r}_j) + (1 - w)E(\tilde{r}_M),$$

$$h(w) = [w^2\sigma_j^2 + (1 - w)^2\sigma_M^2 + 2w(1 - w)\sigma_{jM}]^{1/2},$$

so that

$$E(\tilde{r}_P) = g(w)$$

and

$$\sigma_P = h(w).$$

Deriving the CAPM

Substitute

$$E(\tilde{r}_P) = g(w)$$

and

$$\sigma_P = h(w).$$

into

$$E(\tilde{r}_P) = f(\sigma_P)$$

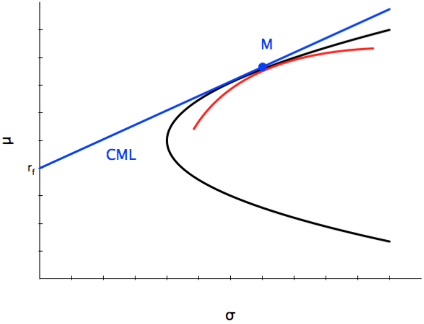
to obtain

$$g(w) = f(h(w))$$

and use the chain rule to compute

$$g'(w) = f'(h(w))h'(w) = f'(\sigma_P)h'(w)$$

Deriving the CAPM



Let $f(\sigma_P)$ be the function defined by $E(\tilde{r}_P) = f(\sigma_P)$ and therefore describing the red curve. Then $f'(\sigma_P)$ is the slope of the curve.

Deriving the CAPM

Hence, to compute $f'(\sigma_P)$, we can rearrange

$$g'(w) = f'(\sigma_P)h'(w)$$

to obtain

$$f'(\sigma_P) = \frac{g'(w)}{h'(w)}$$

and compute $g'(w)$ and $h'(w)$ from the formulas we know.

Deriving the CAPM

$$g(w) = wE(\tilde{r}_j) + (1 - w)E(\tilde{r}_M),$$

implies

$$g'(w) = E(\tilde{r}_j) - E(\tilde{r}_M)$$

Deriving the CAPM

$$h(w) = [w^2\sigma_j^2 + (1-w)^2\sigma_M^2 + 2w(1-w)\sigma_{jM}]^{1/2},$$

implies

$$h'(w) = \frac{1}{2}\left\{\frac{2w\sigma_j^2 - 2(1-w)\sigma_M^2 + 2(1-2w)\sigma_{jM}}{[w^2\sigma_j^2 + (1-w)^2\sigma_M^2 + 2w(1-w)\sigma_{jM}]^{1/2}}\right\}$$

or, a bit more simply,

$$h'(w) = \frac{w\sigma_j^2 - (1-w)\sigma_M^2 + (1-2w)\sigma_{jM}}{[w^2\sigma_j^2 + (1-w)^2\sigma_M^2 + 2w(1-w)\sigma_{jM}]^{1/2}}$$

Deriving the CAPM

$$f'(\sigma_P) = g'(w)/h'(w)$$

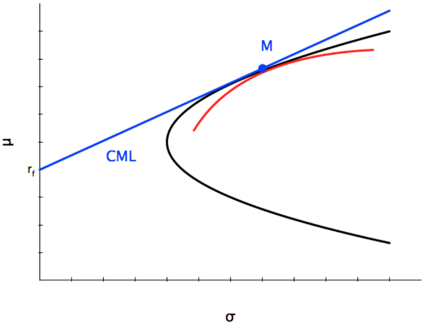
$$g'(w) = E(\tilde{r}_j) - E(\tilde{r}_M)$$

$$h'(w) = \frac{w\sigma_j^2 - (1-w)\sigma_M^2 + (1-2w)\sigma_{jM}}{[w^2\sigma_j^2 + (1-w)^2\sigma_M^2 + 2w(1-w)\sigma_{jM}]^{1/2}}$$

imply

$$\begin{aligned} f'(\sigma_P) &= [E(\tilde{r}_j) - E(\tilde{r}_M)] \\ &\quad \times \frac{[w^2\sigma_j^2 + (1-w)^2\sigma_M^2 + 2w(1-w)\sigma_{jM}]^{1/2}}{w\sigma_j^2 - (1-w)\sigma_M^2 + (1-2w)\sigma_{jM}} \end{aligned}$$

Deriving the CAPM



The red curve is tangent to the CML at M. Hence, $f'(\sigma_P)$ equals the slope of the CML when w=0.

Deriving the CAPM

When $w = 0$,

$$\begin{aligned} f'(\sigma_P) &= [E(\tilde{r}_j) - E(\tilde{r}_M)] \\ &\quad \times \frac{[w^2\sigma_j^2 + (1-w)^2\sigma_M^2 + 2w(1-w)\sigma_{jM}]^{1/2}}{w\sigma_j^2 - (1-w)\sigma_M^2 + (1-2w)\sigma_{jM}} \end{aligned}$$

implies

$$f'(\sigma_P) = \frac{[E(\tilde{r}_j) - E(\tilde{r}_M)]\sigma_M}{\sigma_{jM} - \sigma_M^2}$$

Meanwhile, we know that the slope of the CML is

$$\frac{E(\tilde{r}_M) - r_f}{\sigma_M}$$

Deriving the CAPM

The tangency of the red curve with the CML at M therefore requires

$$\frac{[E(\tilde{r}_j) - E(\tilde{r}_M)]\sigma_M}{\sigma_{jM} - \sigma_M^2} = \frac{E(\tilde{r}_M) - r_f}{\sigma_M}$$

$$E(\tilde{r}_j) - E(\tilde{r}_M) = \frac{[E(\tilde{r}_M) - r_f][\sigma_{jM} - \sigma_M^2]}{\sigma_M^2}$$

$$E(\tilde{r}_j) - E(\tilde{r}_M) = \left(\frac{\sigma_{jM}}{\sigma_M^2}\right)[E(\tilde{r}_M) - r_f] - [E(\tilde{r}_M) - r_f]$$

$$E(\tilde{r}_j) = r_f + \left(\frac{\sigma_{jM}}{\sigma_M^2}\right)[E(\tilde{r}_M) - r_f]$$

Deriving the CAPM

$$E(\tilde{r}_j) = r_f + \left(\frac{\sigma_{jM}}{\sigma_M^2}\right)[E(\tilde{r}_M) - r_f]$$

Let

$$\beta_j = \frac{\sigma_{jM}}{\sigma_M^2}$$

so that this key equation of the CAPM can be written as

$$E(\tilde{r}_j) = r_f + \beta_j[E(\tilde{r}_M) - r_f]$$

where β_j, the "CAPM beta" for asset j, depends on the covariance between the returns on asset j and the market portfolio.

Deriving the CAPM

$$E(\tilde{r}_j) = r_f + \beta_j[E(\tilde{r}_M) - r_f]$$

This equation summarizes a very strong restriction: Given r_f and $E(\tilde{r}_M)$, each stock's expected return (and hence its price today) depends on β_j and only on β_j.

And

$$\beta_j = \frac{\sigma_{jM}}{\sigma_M^2}$$

depends on covariance σ_{jM} not variance σ_j^2.