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FINANCIAL ECONOMICS

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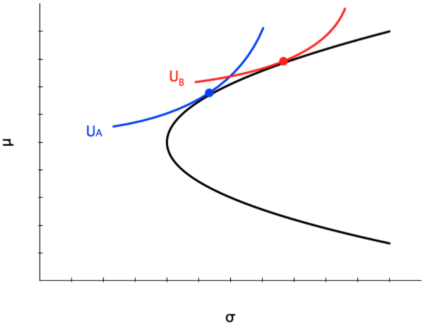
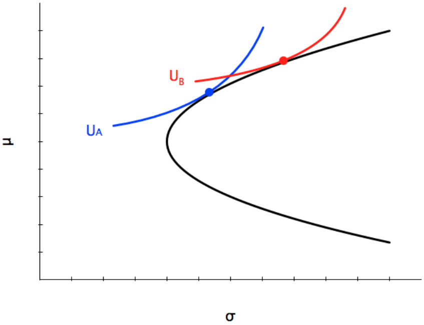
The Efficient Frontier



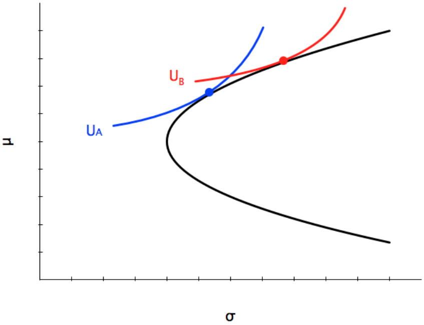
Modern Portfolio Theory implies that all investors choose optimal portfolios along the efficient frontier.

The Efficient Frontier



Fund managers should construct portfolios along the efficient frontier – that are not dominated in mean-variance by any other.

The Efficient Frontier



Individual investors can then choose the portfolio along the efficient frontier that is best suited to their individual levels of risk aversion.

A Separation Theorem

So far, however, our analysis has assumed that there are only risky assets. An additional, quite striking, result emerges when we add a risk free asset to the mix.

This implication was first noted by James Tobin (US, 1918-2002, Nobel Prize 1981) in his paper "Liquidity Preference as Behavior Towards Risk," *Review of Economic Studies* Vol.25 (February 1958): pp.65-86.

A Separation Theorem

Consider, therefore, the larger portfolio formed when an investor allocates the fraction w of his or her initial wealth to a risky asset or to a smaller portfolio of risky assets and the remaining fraction $1 - w$ to a risk free asset with return r_f.

A Separation Theorem

If the risky part of this portfolio has random return $\tilde{r}$, expected return $\mu_r = E(\tilde{r})$, and variance $\sigma_r^2 = E[(\tilde{r} - \mu_r)^2]$ then the larger portfolio has random return $\tilde{r}_P = w\tilde{r} + (1 - w)r_f$ with expected return

$$\mu_P = E[w\tilde{r} + (1 - w)r_f] = w\mu_r + (1 - w)r_f$$

and variance

$$\begin{aligned} \sigma_P^2 &= E[(\tilde{r}_P - \mu_P)^2] \\ &= E\{[w\tilde{r} + (1 - w)r_f - w\mu_r - (1 - w)r_f]^2\} \\ &= E\{[w(\tilde{r} - \mu_r)]^2\} = w^2\sigma_r^2. \end{aligned}$$

A Separation Theorem

The expression for the portfolio's variance

$$\sigma_P^2 = w^2 \sigma_r^2$$

implies (assuming $w \geq 0$)

$$\sigma_P = w\sigma_r$$

and hence

$$w = \frac{\sigma_P}{\sigma_r}.$$

Hence, with σ_r given, a larger share of wealth w allocated to risky assets is associated with a higher standard deviation σ_P for the larger portfolio.

A Separation Theorem

But the expression for the portfolio's expected return

$$\mu_P = w\mu_r + (1 - w)r_f$$

indicates that so long as $\mu_r > r_f$, a higher value of w will yield a higher expected return as well.

What is the trade-off between risk σ_P and expected return μ_P of the mix of risky and riskless assets?

A Separation Theorem

To see, substitute

$$w = \frac{\sigma_P}{\sigma_r}$$

into

$$\mu_P = w\mu_r + (1 - w)r_f$$

to obtain

$$\begin{aligned} \mu_P &= \left(\frac{\sigma_P}{\sigma_r}\right)\mu_r + \left(1 - \frac{\sigma_P}{\sigma_r}\right) r_f \\ &= r_f + \left(\frac{\mu_r - r_f}{\sigma_r}\right)\sigma_P \end{aligned}$$

A Separation Theorem

The expression

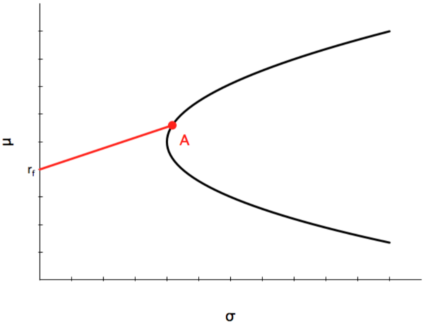
$$\mu_P = r_f + \left(\frac{\mu_r - r_f}{\sigma_r} \right) \sigma_P$$

shows that for portfolios of risky and riskless assets:

1. The relationship between σ_P and μ_P is linear.
2. The slope of the linear relationship is given by the Sharpe ratio, defined here as the "expected excess return" offered by the risky components of the portfolio divided by the standard deviation of the return on that risky component:

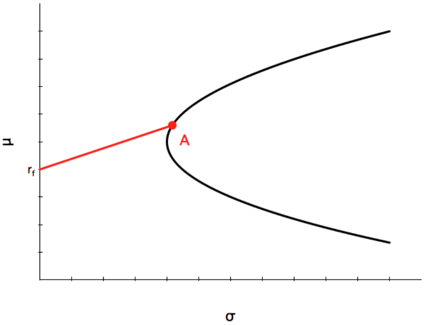
$$\frac{\mu_r - r_f}{\sigma_r} = \frac{E(\tilde{r}) - r_f}{\sigma_r}$$

A Separation Theorem



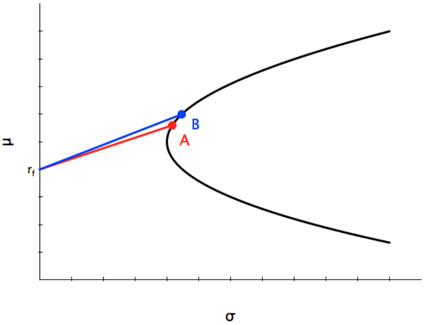
Hence, any investor can combine the risk free asset with risky portfolio A to achieve a combination of expected return and standard deviation along the red line.

A Separation Theorem



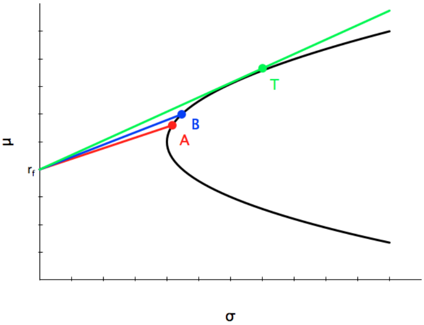
Moving along the red line from r_f to A, the "rise" equals $\mu_A - r_f$ and the "run" equals σ_A. The slope of the red line equals the Sharpe ratio of risky portfolio A.

A Separation Theorem



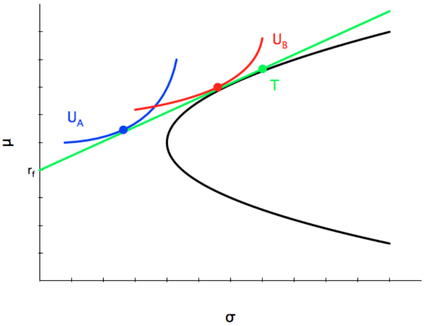
However, any investor with mean-variance utility will prefer some combination of the risk free asset and risky portfolio B to all combinations of the risk free asset and risky portfolio A.

A Separation Theorem



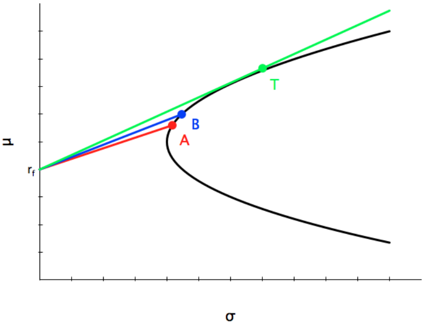
And all investors with mean-variance utility will prefer some combination of the risk free asset and risky portfolio T to any other portfolio.

A Separation Theorem



Investor B is less risk averse than investor A. But both choose some combination of the "tangency portfolio" T and the risk free asset.

A Separation Theorem



Note that the tangency portfolio T can be identified as the portfolio along the efficient frontier of risky assets that has the highest Sharpe ratio.

A Separation Theorem

This is the two-fund theorem or separation theorem implied by Modern Portfolio Theory.

Equity mutual fund managers can all focus on building the unique portfolio that lies along the efficient frontier of risky assets and has the highest Sharpe ratio.

Each individual investor can then tailor his or her own portfolio by choosing the combination of the riskless assets and the risky mutual fund that best suits his or her own aversion to risk.

A Separation Theorem

This is the two-fund theorem or separation theorem implied by Modern Portfolio Theory.

In retirement savings plans, individual investors don't need to choose individual stocks or specialized stock mutual funds.

They only need access to a diversified equity mutual fund run by a manager who successfully maximizes the fund's Sharpe ratio and a money market mutual fund.

A Separation Theorem

Problem Set 13, Questions 1 and 2: An algebraic analysis of the investor's problem.

Suppose the investor's utility function is

$$U(\mu_p, \sigma_p^2) = \mu_p - \left(\frac{A}{2}\right)\sigma_p^2$$

where higher values of A correspond to greater risk aversion.

A Separation Theorem

If this investor chooses between a risky portfolio (perhaps the tangency portfolio) and a risk-free asset

$$\mu_p = r_f + \left(\frac{\mu_r - r_f}{\sigma_r}\right)\sigma_p$$

and

$$\begin{aligned} U(\mu_p, \sigma_p^2) &= \mu_p - \left(\frac{A}{2}\right)\sigma_p^2 \\ &= r_f + \left(\frac{\mu_r - r_f}{\sigma_r}\right)\sigma_p - \left(\frac{A}{2}\right)\sigma_p^2 \end{aligned}$$

A Separation Theorem

The investor's problem

$$\max_{\sigma_p} \; r_f + \left(\frac{\mu_r - r_f}{\sigma_r} \right) \sigma_p - \left(\frac{A}{2} \right) \sigma_p^2$$

Use the FOC

$$\frac{\mu_r - r_f}{\sigma_r} - A\sigma_p^* = 0$$

to find σ_p^* in terms of $\mu_r - r_f$, σ_r, and A. Then use

$$w^* = \frac{\sigma_p^*}{\sigma_r}.$$

to see how w^* depends on $\mu_r - r_f$, σ_r, and A.

A Separation Theorem

w^*, the share optimally allocated to the risky portfolio:

1. Rises when $\mu_r - r_f$ increases.
2. Falls when σ_r or A increases.

Echoing the results from Arrow (1971).

Problem Set 13, Question 3: extension with 2 risky assets.

Strengths and Shortcomings of MPT

We've already considered one shortcoming of the MPT: its mean-variance utility hypothesis must rest on one of two more basic assumptions.

Either utility must be quadratic or asset returns must be normal.

Strengths and Shortcomings of MPT

A second problem involves the estimation or "calibration" of the model's parameters.

With N risky assets, the vector μ of expected returns contains N elements and the matrix Σ of variances and covariances contains $N(N+1)/2$ unique elements. When $N = 100$, for example, there are $100 + (100 \times 101)/2 = 5150$ parameters to estimate! When $N = 500$, there are 125,750 parameters!!!

And to use data from the past to estimate these parameters, one has to assume that past averages and correlations are a reliable guide to the future.

Strengths and Shortcomings of MPT

On the other hand, the MPT teaches us a very important lesson about how individual assets with imperfectly, and especially negatively, correlated returns can be combined into a diversified portfolio to reduce risk.

And the MPT's separation theorem suggests that a retirement savings plan that allows participants to choose between a money market mutual fund and a well-diversified equity fund is fully optimal under certain circumstances and perhaps close enough to optimal more generally.

Strengths and Shortcomings of MPT

Finally, the Capital Asset Pricing Model builds directly on the foundations provided by Modern Portfolio Theory.