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FINANCIAL ECONOMICS

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The Efficient Frontier

$$\mu_P = w\mu_1 + (1-w)\mu_2$$

$$\sigma_P = [w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_1\sigma_2\rho_{12}]^{1/2}$$

In the case with two risky assets, the choice of w simultaneously determines μ_P and σ_P.

The Efficient Frontier

$$\mu_P = w\mu_1 + (1-w)\mu_2$$

$$\sigma_P = [w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_1\sigma_2\rho_{12}]^{1/2}$$

But with more than two risky assets, the portfolio problem takes on an added dimension, since then we can ask: given a target $\mu_P = \bar{\mu}$ for our portfolio's expected return, how can we select $w_1, w_2, \ldots, w_N$ to minimize the standard deviation σ_P?

The Efficient Frontier

$$\mu_P = w\mu_1 + (1-w)\mu_2$$

$$\sigma_P = [w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_1\sigma_2\rho_{12}]^{1/2}$$

Given a target $\mu_P = \bar{\mu}$ for our portfolio's expected return, how can we select $w_1, w_2, \ldots, w_N$ to minimize the standard deviation σ_P? This problem is interesting in its own right, but is also a key step in deriving modern portfolio theory's efficient frontier.

The Efficient Frontier

Consider two portfolios, A and B, with expected returns μ_A and μ_B and standard deviations σ_A and σ_B.

Recall that portfolio A is said to exhibit mean-variance dominance over portfolio B if either

$$\mu_A > \mu_B \text{ and } \sigma_A \leq \sigma_B$$

or

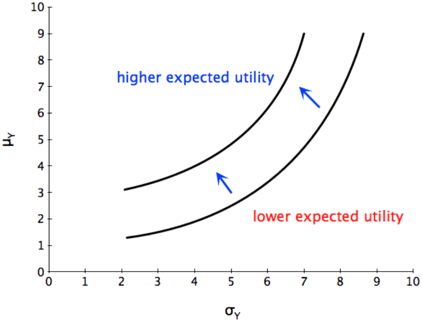
$$\mu_A \geq \mu_B \text{ and } \sigma_A < \sigma_B$$

The Efficient Frontier

Hence, choosing portfolio shares to minimize variance for a given mean will allow us to characterize the efficient frontier:

1. The set of all portfolios that are not mean-variance dominated by any other portfolio.
2. The set of all portfolios that are of potential interest to investors with mean variance utility.
3. The “budget constraint” in Markowitz’s diagram.

The Efficient Frontier



Here are the indifference curves in Markowitz's diagram. Now we want to find out what the constraint looks like when there are more than two risky assets.

The Efficient Frontier

Constructing the efficient frontier brings us back to the concept of mean-variance dominance.

Previously, we detected a problem with this criterion: it fails to distinguish between upside potential and downside risk.

This observation suggests another way of interpreting the idea that MPT and the CAPM "require normally distributed returns."

The symmetry of the normal distribution's bell-shaped curve means that any asset with great upside potential must also have substantial downside risk.

The Efficient Frontier

With three assets, for example, an investor can choose

$$w_1 = \text{share of initial wealth allocated to asset 1}$$

$$w_2 = \text{share of initial wealth allocated to asset 2}$$

$$1 - w_1 - w_2 = \text{share of wealth allocated to asset 3}$$

The Efficient Frontier

Given the choices of w_1 and w_2:

$$\tilde{r}_P = w_1\tilde{r}_1 + w_2\tilde{r}_2 + (1 - w_1 - w_2)\tilde{r}_3$$

$$\mu_P = w_1\mu_1 + w_2\mu_2 + (1 - w_1 - w_2)\mu_3$$

$$\begin{aligned}\sigma_P^2 \quad = \quad & w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + (1 - w_1 - w_2)^2\sigma_3^2 \\ & + 2w_1 w_2\sigma_1\sigma_2\rho_{12} \\ & + 2w_1(1 - w_1 - w_2)\sigma_1\sigma_3\rho_{13} \\ & + 2w_2(1 - w_1 - w_2)\sigma_2\sigma_3\rho_{23}\end{aligned}$$

The Efficient Frontier

Our problem is to solve

$$\min_{w_1,w_2} \sigma_P^2 \text{ subject to } \mu_P = \bar{\mu}$$

for a given value of $\bar{\mu}$.

But since we are more used to solving constrained maximization problems, consider the reformulated, but equivalent, problem:

$$\max_{w_1,w_2} -\sigma_P^2 \text{ subject to } \mu_P = \bar{\mu}$$

The Efficient Frontier

Set up the Lagrangian, using the expressions for σ_P and μ_P derived previously:

$$\begin{aligned} L(w_1, w_2, \lambda) &= -w_1^2\sigma_1^2 - w_2^2\sigma_2^2 - (1 - w_1 - w_2)^2\sigma_3^2 \\ &- 2w_1 w_2 \sigma_1 \sigma_2 \rho_{12} \\ &- 2w_1(1 - w_1 - w_2)\sigma_1\sigma_3\rho_{13} \\ &- 2w_2(1 - w_1 - w_2)\sigma_2\sigma_3\rho_{23} \\ &+ \lambda[w_1\mu_1 + w_2\mu_2 + (1 - w_1 - w_2)\mu_3 - \bar{\mu}] \end{aligned}$$

The Efficient Frontier

$$
\begin{aligned}
L(w_1, w_2, \lambda) \quad = \quad & -w_1^2\sigma_1^2 - w_2^2\sigma_2^2 - (1 - w_1 - w_2)^2\sigma_3^2 \\
& - 2w_1 w_2 \sigma_1 \sigma_2 \rho_{12} \\
& - 2w_1(1 - w_1 - w_2)\sigma_1\sigma_3\rho_{13} \\
& - 2w_2(1 - w_1 - w_2)\sigma_2\sigma_3\rho_{23} \\
& + \lambda[w_1\mu_1 + w_2\mu_2 + (1 - w_1 - w_2)\mu_3 - \bar{\mu}]
\end{aligned}
$$

Most of these objects are data:

$$\mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3, \rho_{12}, \rho_{13}, \rho_{23}$$

And the target $\bar{\mu}$ is given as well.

The Efficient Frontier

PS12, Q2: As in Q1, suppose $\mu_1 = 8$, $\mu_2 = 4$, $\sigma_1 = 8$, and $\sigma_2 = 4$.

Now introduce a third asset, with $\mu_3 = 6$ and $\sigma_3 = 6$.

Assume for simplicity that $\rho_{12} = \rho_{13} = \rho_{23} = 0$.

We can achieve a target expected return $\bar{\mu} = 6$ by investing only in asset 3. The portfolio will then have $\sigma_P = \sigma_3 = 6$.

How much better can we do by choosing portfolio weights optimally? A lot better, even with only 3 assets and zero correlations.

The Efficient Frontier

$$
\begin{aligned}
L(w_1, w_2, \lambda) \quad = \quad & -w_1^2\sigma_1^2 - w_2^2\sigma_2^2 - (1 - w_1 - w_2)^2\sigma_3^2 \\
& - 2w_1 w_2 \sigma_1 \sigma_2 \rho_{12} \\
& - 2w_1(1 - w_1 - w_2)\sigma_1\sigma_3\rho_{13} \\
& - 2w_2(1 - w_1 - w_2)\sigma_2\sigma_3\rho_{23} \\
& + \lambda[w_1\mu_1 + w_2\mu_2 + (1 - w_1 - w_2)\mu_3 - \bar{\mu}]
\end{aligned}
$$

With $\bar{\mu} = 6$, $\mu_1 = 8$, $\mu_2 = 4$, $\mu_3 = 6$, $\sigma_1 = 8$, $\sigma_2 = 4$, $\sigma_3 = 6$, and $\rho_{12} = \rho_{13} = \rho_{23} = 0$:

$$
\begin{aligned}
L(w_1, w_2, \lambda) \quad = \quad & -64w_1^2 - 16w_2^2 - 36(1 - w_1 - w_2)^2 \\
& + \lambda[8w_1 + 4w_2 + 6(1 - w_1 - w_2) - 6]
\end{aligned}
$$

The Efficient Frontier

$$\begin{aligned} L(w_1, w_2, \lambda) &= -64w_1^2 - 16w_2^2 - 36(1 - w_1 - w_2)^2 \\ &\quad + \lambda[8w_1 + 4w_2 + 6(1 - w_1 - w_2) - 6] \end{aligned}$$

FOC for w_1:

$$-128w_1^* + 72(1 - w_1^* - w_2^*) + \lambda^*(8 - 6) = 0$$

FOC for w_2:

$$-32w_2^* + 72(1 - w_1^* - w_2^*) + \lambda^*(4 - 6) = 0$$

Constraint:

$$8w_1^* + 4w_2^* + 6(1 - w_1^* - w_2^*) = 6$$

The Efficient Frontier

FOC for w_1:

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$$-32w_2^* + 72(1 - w_1^* - w_2^*) + \lambda^*(4 - 6) = 0$$

Constraint:

$$8w_1^* + 4w_2^* + 6(1 - w_1^* - w_2^*) = 6$$

Three linear equations in three unknowns: w_1^*, w_2^*, and λ^*.

The Efficient Frontier

Start with the constraint:

$$8w_1^* + 4w_2^* + 6(1 - w_1^* - w_2^*) = 6$$

$$2w_1^* - 2w_2^* = 0$$

$$w_1^* = w_2^*$$

Since $\mu_1 = 8$ and $\mu_2 = 4$, maintaining the target expected return $\mu_P = 6$ requires allocating equal shares to assets 1 and 2.

The Efficient Frontier

Substitute

$$w^* = w_1^* = w_2^*$$

Into the FOC for w_1:

$$-128w_1^* + 72(1 - w_1^* - w_2^*) + \lambda^*(8 - 6) = 0$$

$$-128w^* + 72(1 - 2w^*) + 2\lambda^* = 0$$

and the FOC for w_2:

$$-32w_2^* + 72(1 - w_1^* - w_2^*) + \lambda^*(4 - 6) = 0$$

$$-32w^* + 72(1 - 2w^*) - 2\lambda^* = 0$$

The Efficient Frontier

Solve for w^* by elimination:

$$-128w^* + 72(1 - 2w^*) + 2\lambda^* = 0$$

$$-32w^* + 72(1 - 2w^*) - 2\lambda^* = 0$$

$$-160w^* + 144(1 - 2w^*) = 0$$

The Efficient Frontier

$$-160w^* + 144(1 - 2w^*) = 0$$

After you find the numerical values of $w_1^* = w^*$, $w_2^* = w^*$, and $w_3^* = 1 - w_1^* - w_2^* = 1 - 2w^*$, compute

$$\sigma_P = (64w_1^{*2} + 16w_2^{*2} + 36w_3^{*2})^{1/2}$$

It will be much smaller than 6. Optimal portfolio allocation yields a substantial reduction in risk while still maintaining the expected return of $\bar{\mu} = 6$ percent.

The Efficient Frontier

$$\begin{aligned} L(w_1, w_2, \lambda) &= -w_1^2\sigma_1^2 - w_2^2\sigma_2^2 - (1 - w_1 - w_2)^2\sigma_3^2 \\ &- 2w_1 w_2 \sigma_1\sigma_2\rho_{12} \\ &- 2w_1(1 - w_1 - w_2)\sigma_1\sigma_3\rho_{13} \\ &- 2w_2(1 - w_1 - w_2)\sigma_2\sigma_3\rho_{23} \\ &+ \lambda[w_1\mu_1 + w_2\mu_2 + (1 - w_1 - w_2)\mu_3 - \bar{\mu}] \end{aligned}$$

First-order condition for w_1:

$$\begin{aligned} 0 &= -2w_1^*\sigma_1^2 + 2(1 - w_1^* - w_2^*)\sigma_3^2 - 2w_2^*\sigma_1\sigma_2\rho_{12} \\ &- 2(1 - w_1^* - w_2^*)\sigma_1\sigma_3\rho_{13} + 2w_1^*\sigma_1\sigma_3\rho_{13} \\ &+ 2w_2^*\sigma_2\sigma_3\rho_{23} + \lambda^*\mu_1 - \lambda^*\mu_3 \end{aligned}$$

The Efficient Frontier

$$\begin{aligned} L(w_1, w_2, \lambda) &= -w_1^2\sigma_1^2 - w_2^2\sigma_2^2 - (1 - w_1 - w_2)^2\sigma_3^2 \\ &- 2w_1 w_2 \sigma_1\sigma_2\rho_{12} \\ &- 2w_1(1 - w_1 - w_2)\sigma_1\sigma_3\rho_{13} \\ &- 2w_2(1 - w_1 - w_2)\sigma_2\sigma_3\rho_{23} \\ &+ \lambda[w_1\mu_1 + w_2\mu_2 + (1 - w_1 - w_2)\mu_3 - \bar{\mu}] \end{aligned}$$

First-order condition for w_2:

$$\begin{aligned} 0 &= -2w_2^*\sigma_2^2 + 2(1 - w_1^* - w_2^*)\sigma_3^2 - 2w_1^*\sigma_1\sigma_2\rho_{12} \\ &+ 2w_1^*\sigma_1\sigma_3\rho_{13} - 2(1 - w_1^* - w_2^*)\sigma_2\sigma_3\rho_{23} \\ &+ 2w_2^*\sigma_2\sigma_3\rho_{23} + \lambda^*\mu_2 - \lambda^*\mu_3 \end{aligned}$$

The Efficient Frontier

The two first-order conditions and the constraint

$$\begin{aligned} 0 &= -2w_1^*\sigma_1^2 + 2(1 - w_1^* - w_2^*)\sigma_3^2 - 2w_2^*\sigma_1\sigma_2\rho_{12} \\ &\quad - 2(1 - w_1^* - w_2^*)\sigma_1\sigma_3\rho_{13} + 2w_1^*\sigma_1\sigma_3\rho_{13} \\ &\quad + 2w_2^*\sigma_2\sigma_3\rho_{23} + \lambda^*\mu_1 - \lambda^*\mu_3 \end{aligned}$$

$$\begin{aligned} 0 &= -2w_2^*\sigma_2^2 + 2(1 - w_1^* - w_2^*)\sigma_3^2 - 2w_1^*\sigma_1\sigma_2\rho_{12} \\ &\quad + 2w_1^*\sigma_1\sigma_3\rho_{13} - 2(1 - w_1^* - w_2^*)\sigma_2\sigma_3\rho_{23} \\ &\quad + 2w_2^*\sigma_2\sigma_3\rho_{23} + \lambda^*\mu_2 - \lambda^*\mu_3 \end{aligned}$$

$$w_1^*\mu_1 + w_2^*\mu_2 + (1 - w_1^* - w_2^*)\mu_3 = \bar{\mu}$$

form a system of three equations in the three unknowns: w_1^*, w_2^*, and λ^*.

The Efficient Frontier

Moreover, the equations are linear in the unknowns w_1^*, w_2^*, and λ^*:

$$
\begin{aligned}
0 &= -2w_1^*\sigma_1^2 + 2(1-w_1^*-w_2^*)\sigma_3^2 - 2w_2^*\sigma_1\sigma_2\rho_{12} \\
&\quad - 2(1-w_1^*-w_2^*)\sigma_1\sigma_3\rho_{13} + 2w_1^*\sigma_1\sigma_3\rho_{13} \\
&\quad + 2w_2^*\sigma_2\sigma_3\rho_{23} + \lambda^*\mu_1 - \lambda^*\mu_3
\end{aligned}
$$

$$
\begin{aligned}
0 &= -2w_2^*\sigma_2^2 + 2(1-w_1^*-w_2^*)\sigma_3^2 - 2w_1^*\sigma_1\sigma_2\rho_{12} \\
&\quad + 2w_1^*\sigma_1\sigma_3\rho_{13} - 2(1-w_1^*-w_2^*)\sigma_2\sigma_3\rho_{23} \\
&\quad + 2w_2^*\sigma_2\sigma_3\rho_{23} + \lambda^*\mu_2 - \lambda^*\mu_3
\end{aligned}
$$

$$
w_1^*\mu_1 + w_2^*\mu_2 + (1-w_1^*-w_2^*)\mu_3 = \bar{\mu}
$$

Given specific values for μ_1, μ_2, μ_3, σ_1, σ_2, σ_3, ρ_{12}, ρ_{13}, ρ_{23}, and $\bar{\mu}$ they can be solved quite easily.

The Efficient Frontier

In linear algebra, a vector is just a column of numbers. With $N \geq 3$ assets, you can organize the portfolio shares and expected returns into a vectors:

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix} \text{ and } \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{bmatrix}$$

where

$$w_1 + w_2 + \ldots + w_N = 1$$

Also in linear algebra, the transpose of a vector just reorganizes the column as a row; for example:

$$w' = \begin{bmatrix} w_1 & w_2 & \ldots & w_N \end{bmatrix}$$

The Efficient Frontier

Meanwhile, the variances and covariances can be organized into a matrix – a collection of rows and columns:

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{12} & \dots & \sigma_1\sigma_N\rho_{1N} \\ \sigma_1\sigma_2\rho_{12} & \sigma_2^2 & \dots & \sigma_2\sigma_N\rho_{2N} \\ \vdots & \vdots & \dots & \vdots \\ \sigma_1\sigma_N\rho_{1N} & \sigma_2\sigma_N\rho_{2N} & \dots & \sigma_N^2 \end{bmatrix}$$

The Efficient Frontier

Using the rules from linear algebra for multiplying vectors and matrices, the expected return on any portfolio with shares in the vector w is

$$\mu' w$$

and the variance of the random return on the portfolio is

$$w' \Sigma w.$$

Hence, the problem of minimizing the variance for a given mean can be written compactly as

$$\max_{w} -w' \Sigma w \text{ subject to } \mu' w = \bar{\mu} \text{ and } \ell' w = 1$$

where ℓ is a vector of N ones.

The Efficient Frontier

$$\max_w -w'\Sigma w \text{ subject to } \mu' w = \bar{\mu} \text{ and } \ell' w = 1$$

Problems of this form are called quadratic programming problems and can be solved very quickly on a computer even when the number of assets N is large.

We can also add more constraints, such as $w_i \geq 0$, ruling out short sales.

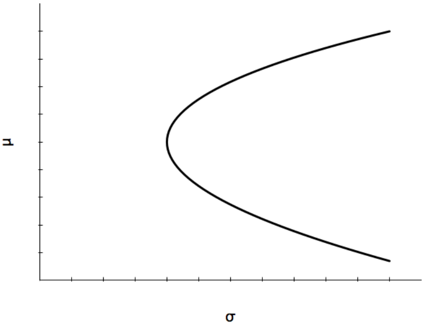
The Efficient Frontier

Going back to the case with three assets, once the optimal shares w_1^* and w_2^* have been found, the minimized standard deviation can be computed using the general formula

$$\begin{aligned}\sigma_P^2 &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + (1 - w_1 - w_2)^2\sigma_3^2 \\ &+ 2w_1 w_2 \sigma_1\sigma_2\rho_{12} \\ &+ 2w_1(1 - w_1 - w_2)\sigma_1\sigma_3\rho_{13} \\ &+ 2w_2(1 - w_1 - w_2)\sigma_2\sigma_3\rho_{23}\end{aligned}$$

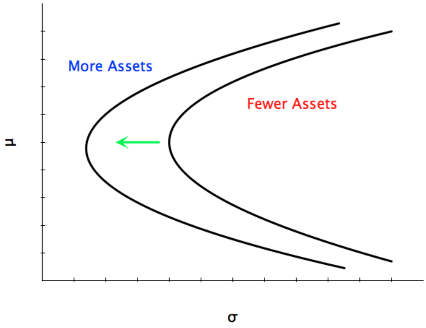
Doing this for various values of $\bar{\mu}$ allows us to trace out the minimum variance frontier.

The Efficient Frontier



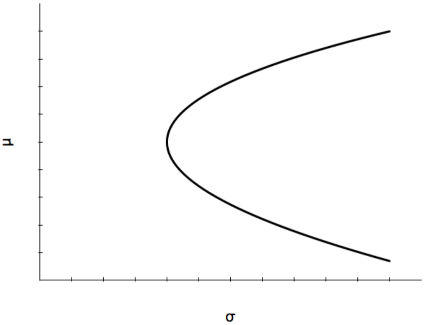
Tracing out the minimized σ_p for each value of $\mu_p = \bar{\mu}$ produces the minimum variance frontier.

The Efficient Frontier



Adding assets shifts the minimum variance frontier to the left, as opportunities for diversification are enhanced.

The Efficient Frontier

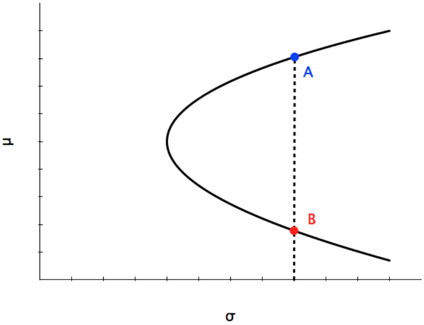


However, the minimum variance frontier retains its sideways parabolic shape.

The Efficient Frontier

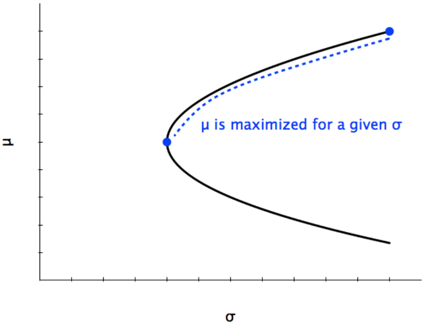
The minimum variance frontier traces out the minimized variance or standard deviation for each required mean return.

The Efficient Frontier



But portfolio A exhibits mean-variance dominance over portfolio B, since it offers a higher expected return with the same standard deviation.

The Efficient Frontier



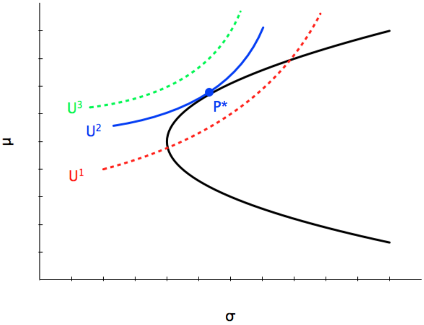
Hence, the efficient frontier extends only along the top arm of the minimum variance frontier.

The Efficient Frontier

Recall that any of the following assumptions imply that indifference curves in this $\sigma - \mu$ diagram slope upward and are convex:

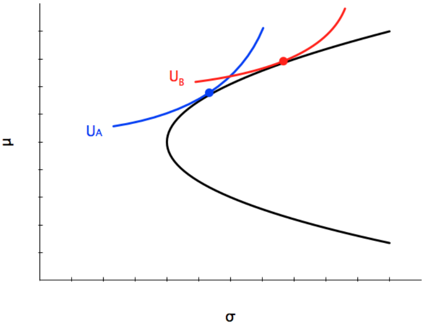
1. Risks are small enough to justify a second-order Taylor approximation to any increasing and concave Bernoulli utility function within the vN-M expected utility framework
2. Investors have vN-M expected utility with quadratic Bernoulli utility functions
3. Asset returns are normally distributed and investors have vN-M expected utility with increasing and concave Bernoulli utility functions

The Efficient Frontier



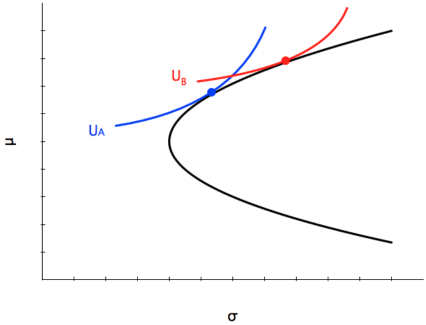
Portfolios along U^1 are suboptimal. Portfolios along U^3 are infeasible. Portfolio P^*, located where U^2 is tangent to the efficient frontier, is optimal.

The Efficient Frontier



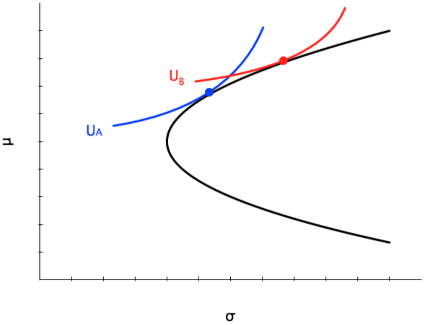
Investor B is less risk averse than investor A. But both choose portfolios along the efficient frontier.

The Efficient Frontier



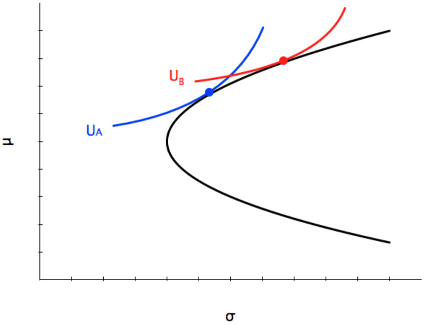
Thus, the mean-variance utility hypothesis built into Modern Portfolio Theory implies that all investors choose optimal portfolios along the efficient frontier.

The Efficient Frontier



Fund managers should construct portfolios along the efficient frontier – that are not dominated in mean-variance by any other.

The Efficient Frontier



Individual investors can then choose the portfolio along the efficient frontier that is best suited to their individual levels of risk aversion.