The Efficient Frontier

\[ \mu_P = w \mu_1 + (1 - w) \mu_2 \]

\[ \sigma_P = \left[ w^2 \sigma_1^2 + (1 - w)^2 \sigma_2^2 + 2w(1 - w)\sigma_1 \sigma_2 \rho_{12} \right]^{1/2} \]

In the case with two risky assets, the choice of \( w \) simultaneously determines \( \mu_P \) and \( \sigma_P \).
The Efficient Frontier

$$\mu_P = w\mu_1 + (1 - w)\mu_2$$

$$\sigma_P = \left[w^2\sigma_1^2 + (1 - w)^2\sigma_2^2 + 2w(1 - w)\sigma_1\sigma_2\rho_{12}\right]^{1/2}$$

But with more than two risky assets, the portfolio problem takes on an added dimension, since then we can ask: given a target $$\mu_P = \bar{\mu}$$ for our portfolio’s expected return, how can we select $$w_1, w_2, \ldots, w_N$$ to minimize the standard deviation $$\sigma_P$$?
The Efficient Frontier

\[ \mu_P = w \mu_1 + (1 - w) \mu_2 \]

\[ \sigma_P = \left[ w^2 \sigma_1^2 + (1 - w)^2 \sigma_2^2 + 2w(1 - w)\sigma_1 \sigma_2 \rho_{12} \right]^{1/2} \]

Given a target \( \mu_P = \bar{\mu} \) for our portfolio’s expected return, how can we select \( w_1, w_2, \ldots, w_N \) to minimize the standard deviation \( \sigma_P \)? This problem is interesting in its own right, but is also a key step in deriving modern portfolio theory’s efficient frontier.
The Efficient Frontier

Consider two portfolios, $A$ and $B$, with expected returns $\mu_A$ and $\mu_B$ and standard deviations $\sigma_A$ and $\sigma_B$.

Recall that portfolio $A$ is said to exhibit mean-variance dominance over portfolio $B$ if either

$$\mu_A > \mu_B \text{ and } \sigma_A \leq \sigma_B$$

or

$$\mu_A \geq \mu_B \text{ and } \sigma_A < \sigma_B$$
Hence, choosing portfolio shares to minimize variance for a given mean will allow us to characterize the efficient frontier:

1. The set of all portfolios that are not mean-variance dominated by any other portfolio.
2. The set of all portfolios that are of potential interest to investors with mean variance utility.
3. The “budget constraint” in Markowitz’s diagram.
Here are the indifference curves in Markowitz's diagram. Now we want to find out what the constraint looks like when there are more than two risky assets.
The Efficient Frontier

Constructing the efficient frontier brings us back to the concept of mean-variance dominance.

Previously, we detected a problem with this criterion: it fails to distinguish between upside potential and downside risk.

This observation suggests another way of interpreting the idea that MPT and the CAPM “require normally distributed returns.”

The symmetry of the normal distribution’s bell-shaped curve means that any asset with great upside potential must also have substantial downside risk.
The Efficient Frontier

With three assets, for example, an investor can choose

\[ w_1 = \text{share of initial wealth allocated to asset 1} \]

\[ w_2 = \text{share of initial wealth allocated to asset 2} \]

\[ 1 - w_1 - w_2 = \text{share of wealth allocated to asset 3} \]
The Efficient Frontier

Given the choices of $w_1$ and $w_2$:

$$
\tilde{r}_P = w_1 \tilde{r}_1 + w_2 \tilde{r}_2 + (1 - w_1 - w_2) \tilde{r}_3
$$

$$
\mu_P = w_1 \mu_1 + w_2 \mu_2 + (1 - w_1 - w_2) \mu_3
$$

$$
\sigma^2_P = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + (1 - w_1 - w_2)^2 \sigma_3^2 \\
+ 2w_1 w_2 \sigma_1 \sigma_2 \rho_{12} \\
+ 2w_1 (1 - w_1 - w_2) \sigma_1 \sigma_3 \rho_{13} \\
+ 2w_2 (1 - w_1 - w_2) \sigma_2 \sigma_3 \rho_{23}
$$
The Efficient Frontier

Our problem is to solve

$$\min_{w_1, w_2} \sigma_P^2 \quad \text{subject to} \quad \mu_P = \bar{\mu}$$

for a given value of $\bar{\mu}$.

But since we are more used to solving constrained maximization problems, consider the reformulated, but equivalent, problem:

$$\max_{w_1, w_2} -\sigma_P^2 \quad \text{subject to} \quad \mu_P = \bar{\mu}$$
The Efficient Frontier

Set up the Lagrangian, using the expressions for \( \sigma_P \) and \( \mu_P \) derived previously:

\[
L(w_1, w_2, \lambda) = -w_1^2 \sigma_1^2 - w_2^2 \sigma_2^2 - (1 - w_1 - w_2)^2 \sigma_3^2 \\
- 2w_1 w_2 \sigma_1 \sigma_2 \rho_{12} \\
- 2w_1(1 - w_1 - w_2) \sigma_1 \sigma_3 \rho_{13} \\
- 2w_2(1 - w_1 - w_2) \sigma_2 \sigma_3 \rho_{23} \\
+ \lambda [w_1 \mu_1 + w_2 \mu_2 + (1 - w_1 - w_2) \mu_3 - \bar{\mu}] 
\]
The Efficient Frontier

\[ L(w_1, w_2, \lambda) = -w_1^2 \sigma_1^2 - w_2^2 \sigma_2^2 - (1 - w_1 - w_2)^2 \sigma_3^2 \]
\[ - 2w_1 w_2 \sigma_1 \sigma_2 \rho_{12} \]
\[ - 2w_1 (1 - w_1 - w_2) \sigma_1 \sigma_3 \rho_{13} \]
\[ - 2w_2 (1 - w_1 - w_2) \sigma_2 \sigma_3 \rho_{23} \]
\[ + \lambda [w_1 \mu_1 + w_2 \mu_2 + (1 - w_1 - w_2) \mu_3 - \bar{\mu}] \]

Most of these objects are data:

\[ \mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3, \rho_{12}, \rho_{13}, \rho_{23} \]

And the target \( \bar{\mu} \) is given as well.
PS12, Q2: As in Q1, suppose \( \mu_1 = 8, \mu_2 = 4, \sigma_1 = 8, \) and \( \sigma_2 = 4. \)

Now introduce a third asset, with \( \mu_3 = 6 \) and \( \sigma_3 = 6. \)

Assume for simplicity that \( \rho_{12} = \rho_{13} = \rho_{23} = 0. \)

We can achieve a target expected return \( \bar{\mu} = 6 \) by investing only in asset 3. The portfolio will then have \( \sigma_P = \sigma_3 = 6. \)

How much better can we do by choosing portfolio weights optimally? A lot better, even with only 3 assets and zero correlations.
The Efficient Frontier

\[
L(w_1, w_2, \lambda) = -w_1^2 \sigma_1^2 - w_2^2 \sigma_2^2 - (1 - w_1 - w_2)^2 \sigma_3^2 \\
- 2w_1 w_2 \sigma_1 \sigma_2 \rho_{12} \\
- 2w_1 (1 - w_1 - w_2) \sigma_1 \sigma_3 \rho_{13} \\
- 2w_2 (1 - w_1 - w_2) \sigma_2 \sigma_3 \rho_{23} \\
+ \lambda [w_1 \mu_1 + w_2 \mu_2 + (1 - w_1 - w_2) \mu_3 - \bar{\mu}]
\]

With \( \bar{\mu} = 6, \mu_1 = 8, \mu_2 = 4, \mu_3 = 6, \sigma_1 = 8, \sigma_2 = 4, \sigma_3 = 6, \)
and \( \rho_{12} = \rho_{13} = \rho_{23} = 0: \)

\[
L(w_1, w_2, \lambda) = -64w_1^2 - 16w_2^2 - 36(1 - w_1 - w_2)^2 \\
+ \lambda [8w_1 + 4w_2 + 6(1 - w_1 - w_2) - 6]
\]
The Efficient Frontier

\[ L(w_1, w_2, \lambda) = -64w_1^2 - 16w_2^2 - 36(1 - w_1 - w_2)^2 \\
+ \lambda[8w_1 + 4w_2 + 6(1 - w_1 - w_2) - 6] \]

FOC for \( w_1 \):

\[-128w_1^* + 72(1 - w_1^* - w_2^*) + \lambda^*(8 - 6) = 0 \]

FOC for \( w_2 \):

\[-32w_2^* + 72(1 - w_1^* - w_2^*) + \lambda^*(4 - 6) = 0 \]

Constraint:

\[8w_1^* + 4w_2^* + 6(1 - w_1^* - w_2^*) = 6 \]
The Efficient Frontier

FOC for $w_1$:

$$-128w_1^* + 72(1 - w_1^* - w_2^*) + \lambda^*(8 - 6) = 0$$

FOC for $w_2$:

$$-32w_2^* + 72(1 - w_1^* - w_2^*) + \lambda^*(4 - 6) = 0$$

Constraint:

$$8w_1^* + 4w_2^* + 6(1 - w_1^* - w_2^*) = 6$$

Three linear equations in three unknowns: $w_1^*$, $w_2^*$, and $\lambda^*$. 
The Efficient Frontier

Start with the constraint:

\[ 8w_1^* + 4w_2^* + 6(1 - w_1^* - w_2^*) = 6 \]

\[ 2w_1^* - 2w_2^* = 0 \]

\[ w_1^* = w_2^* \]

Since \( \mu_1 = 8 \) and \( \mu_2 = 4 \), maintaining the target expected return \( \mu_P = 6 \) requires allocating equal shares to assets 1 and 2.
The Efficient Frontier

Substitute

\[ w^* = w_1^* = w_2^* \]

Into the FOC for \( w_1 \):

\[ -128w_1^* + 72(1 - w_1^* - w_2^*) + \lambda^*(8 - 6) = 0 \]

\[ -128w^* + 72(1 - 2w^*) + 2\lambda^* = 0 \]

and the FOC for \( w_2 \):

\[ -32w_2^* + 72(1 - w_1^* - w_2^*) + \lambda^*(4 - 6) = 0 \]

\[ -32w^* + 72(1 - 2w^*) - 2\lambda^* = 0 \]

Solve for $w^*$ by elimination:

$$-128w^* + 72(1 - 2w^*) + 2\lambda^* = 0$$

$$-32w^* + 72(1 - 2w^*) - 2\lambda^* = 0$$

$$-160w^* + 144(1 - 2w^*) = 0$$
The Efficient Frontier

\[-160w^* + 144(1 - 2w^*) = 0\]

After you find the numerical values of $w_1^* = w^*$, $w_2^* = w^*$, and $w_3^* = 1 - w_1^* - w_2^* = 1 - 2w^*$, compute

$$\sigma_P = (64w_1^{*2} + 16w_2^{*2} + 36w_3^{*2})^{1/2}$$

It will be much smaller than 6. Optimal portfolio allocation yields a substantial reduction in risk while still maintaining the expected return of $\bar{\mu} = 6$ percent.
The Efficient Frontier

\[
L(w_1, w_2, \lambda) = -w_1^2 \sigma_1^2 - w_2^2 \sigma_2^2 - (1 - w_1 - w_2)^2 \sigma_3^2 \\
- 2w_1 w_2 \sigma_1 \sigma_2 \rho_{12} \\
- 2w_1 (1 - w_1 - w_2) \sigma_1 \sigma_3 \rho_{13} \\
- 2w_2 (1 - w_1 - w_2) \sigma_2 \sigma_3 \rho_{23} \\
+ \lambda [w_1 \mu_1 + w_2 \mu_2 + (1 - w_1 - w_2) \mu_3 - \bar{\mu}]
\]

First-order condition for \( w_1 \):

\[
0 = -2w_1^* \sigma_1^2 + 2(1 - w_1^* - w_2^*) \sigma_3^2 - 2w_2^* \sigma_1 \sigma_2 \rho_{12} \\
- 2(1 - w_1^* - w_2^*) \sigma_1 \sigma_3 \rho_{13} + 2w_1^* \sigma_1 \sigma_3 \rho_{13} \\
+ 2w_2^* \sigma_2 \sigma_3 \rho_{23} + \lambda^* \mu_1 - \lambda^* \mu_3
\]
The Efficient Frontier

\[ L(w_1, w_2, \lambda) = -w_1^2 \sigma_1^2 - w_2^2 \sigma_2^2 - (1 - w_1 - w_2)^2 \sigma_3^2 \]
\[ - 2w_1 w_2 \sigma_1 \sigma_2 \rho_{12} \]
\[ - 2w_1 (1 - w_1 - w_2) \sigma_1 \sigma_3 \rho_{13} \]
\[ - 2w_2 (1 - w_1 - w_2) \sigma_2 \sigma_3 \rho_{23} \]
\[ + \lambda [w_1 \mu_1 + w_2 \mu_2 + (1 - w_1 - w_2) \mu_3 - \bar{\mu}] \]

First-order condition for \( w_2 \):

\[ 0 = -2w_2^* \sigma_2^2 + 2(1 - w_1^* - w_2^*) \sigma_3^2 - 2w_1^* \sigma_1 \sigma_2 \rho_{12} \]
\[ + 2w_1^* \sigma_1 \sigma_3 \rho_{13} - 2(1 - w_1^* - w_2^*) \sigma_2 \sigma_3 \rho_{23} \]
\[ + 2w_2^* \sigma_2 \sigma_3 \rho_{23} + \lambda^* \mu_2 - \lambda^* \mu_3 \]
The Efficient Frontier

The two first-order conditions and the constraint

\[
0 = -2w_1^* \sigma_1^2 + 2(1 - w_1^* - w_2^*) \sigma_3^2 - 2w_2^* \sigma_1 \sigma_2 \rho_{12} \\
- 2(1 - w_1^* - w_2^*) \sigma_1 \sigma_3 \rho_{13} + 2w_1^* \sigma_1 \sigma_3 \rho_{13} \\
+ 2w_2^* \sigma_2 \sigma_3 \rho_{23} + \lambda^* \mu_1 - \lambda^* \mu_3
\]

\[
0 = -2w_2^* \sigma_2^2 + 2(1 - w_1^* - w_2^*) \sigma_3^2 - 2w_1^* \sigma_1 \sigma_2 \rho_{12} \\
+ 2w_1^* \sigma_1 \sigma_3 \rho_{13} - 2(1 - w_1^* - w_2^*) \sigma_2 \sigma_3 \rho_{23} \\
+ 2w_2^* \sigma_2 \sigma_3 \rho_{23} + \lambda^* \mu_2 - \lambda^* \mu_3
\]

\[
w_1^* \mu_1 + w_2^* \mu_2 + (1 - w_1^* - w_2^*) \mu_3 = \bar{\mu}
\]

form a system of three equations in the three unknowns: \( w_1^* \), \( w_2^* \), and \( \lambda^* \).
The Efficient Frontier
Moreover, the equations are linear in the unknowns $w_1^*$, $w_2^*$, and $\lambda^*$:

$$0 = -2w_1^*\sigma_1^2 + 2(1 - w_1^* - w_2^*)\sigma_3^2 - 2w_2^*\sigma_1\sigma_2\rho_{12}$$
$$+ 2(1 - w_1^* - w_2^*)\sigma_1\sigma_3\rho_{13} + 2w_1^*\sigma_1\sigma_3\rho_{13}$$
$$+ 2w_2^*\sigma_2\sigma_3\rho_{23} + \lambda^*\mu_1 - \lambda^*\mu_3$$

$$0 = -2w_2^*\sigma_2^2 + 2(1 - w_1^* - w_2^*)\sigma_3^2 - 2w_1^*\sigma_1\sigma_2\rho_{12}$$
$$+ 2w_1^*\sigma_1\sigma_3\rho_{13} - 2(1 - w_1^* - w_2^*)\sigma_2\sigma_3\rho_{23}$$
$$+ 2w_2^*\sigma_2\sigma_3\rho_{23} + \lambda^*\mu_2 - \lambda^*\mu_3$$

$$w_1^*\mu_1 + w_2^*\mu_2 + (1 - w_1^* - w_2^*)\mu_3 = \bar{\mu}$$

Given specific values for $\mu_1$, $\mu_2$, $\mu_3$, $\sigma_1$, $\sigma_2$, $\sigma_3$, $\rho_{12}$, $\rho_{13}$, $\rho_{23}$, and $\bar{\mu}$ they can be solved quite easily.
The Efficient Frontier
In linear algebra, a vector is just a column of numbers. With \( N \geq 3 \) assets, you can organize the portfolio shares and expected returns into a vectors:

\[
w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix} \quad \text{and} \quad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{bmatrix}
\]

where

\[ w_1 + w_2 + \ldots + w_N = 1 \]

Also in linear algebra, the transpose of a vector just reorganizes the column as a row; for example:

\[
w' = [w_1 \ w_2 \ \ldots \ \ w_N]
\]
Meanwhile, the variances and covariances can be organized into a matrix – a collection of rows and columns:

\[ \Sigma = \begin{bmatrix}
\sigma_1^2 & \sigma_1 \sigma_2 \rho_{12} & \ldots & \sigma_1 \sigma_N \rho_{1N} \\
\sigma_1 \sigma_2 \rho_{12} & \sigma_2^2 & \ldots & \sigma_2 \sigma_N \rho_{2N} \\
\vdots & \vdots & \ldots & \vdots \\
\sigma_1 \sigma_N \rho_{1N} & \sigma_2 \sigma_N \rho_{2N} & \ldots & \sigma_N^2
\end{bmatrix} \]
The Efficient Frontier

Using the rules from linear algebra for multiplying vectors and matrices, the expected return on any portfolio with shares in the vector $w$ is

$$\mu'w$$

and the variance of the random return on the portfolio is

$$w'\Sigma w.$$ 

Hence, the problem of minimizing the variance for a given mean can be written compactly as

$$\max_w -w'\Sigma w \text{ subject to } \mu'w = \bar{\mu} \text{ and } \ell'w = 1$$

where $\ell$ is a vector of $N$ ones.
The Efficient Frontier

\[
\max_{\mathbf{w}} - \mathbf{w}'\Sigma\mathbf{w} \quad \text{subject to} \quad \mathbf{\mu}'\mathbf{w} = \bar{\mu} \quad \text{and} \quad \ell'\mathbf{w} = 1
\]

Problems of this form are called \textit{quadratic programming problems} and can be solved very quickly on a computer even when the number of assets \( N \) is large.

We can also add more constraints, such as \( w_i \geq 0 \), ruling out short sales.
Going back to the case with three assets, once the optimal
shares $w_1^*$ and $w_2^*$ have been found, the minimized standard
deviation can be computed using the general formula

$$
\sigma_P^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + (1 - w_1 - w_2)^2 \sigma_3^2 \\
+ 2w_1 w_2 \sigma_1 \sigma_2 \rho_{12} \\
+ 2w_1 (1 - w_1 - w_2) \sigma_1 \sigma_3 \rho_{13} \\
+ 2w_2 (1 - w_1 - w_2) \sigma_2 \sigma_3 \rho_{23}
$$

Doing this for various values of $\bar{\mu}$ allows us to trace out the
minimum variance frontier.
The Efficient Frontier

Tracing out the minimized $\sigma_p$ for each value of $\mu_p = \bar{\mu}$ produces the minimum variance frontier.
Adding assets shifts the minimum variance frontier to the left, as opportunities for diversification are enhanced.
The Efficient Frontier

However, the minimum variance frontier retains its sideways parabolic shape.
The Efficient Frontier

The minimum variance frontier traces out the minimized variance or standard deviation for each required mean return.
But portfolio A exhibits mean-variance dominance over portfolio B, since it offers a higher expected return with the same standard deviation.
Hence, the efficient frontier extends only along the top arm of the minimum variance frontier.
The Efficient Frontier

Recall that any of the following assumptions imply that indifference curves in this $\sigma - \mu$ diagram slope upward and are convex:

1. Risks are small enough to justify a second-order Taylor approximation to any increasing and concave Bernoulli utility function within the vN-M expected utility framework.

2. Investors have vN-M expected utility with quadratic Bernoulli utility functions.

3. Asset returns are normally distributed and investors have vN-M expected utility with increasing and concave Bernoulli utility functions.
The Efficient Frontier

Portfolios along $U^1$ are suboptimal. Portfolios along $U^3$ are infeasible. Portfolio $P^*$, located where $U^2$ is tangent to the efficient frontier, is optimal.
Investor B is less risk averse than investor A. But both choose portfolios along the efficient frontier.
Thus, the mean-variance utility hypothesis built into Modern Portfolio Theory implies that all investors choose optimal portfolios along the efficient frontier.
The Efficient Frontier

Fund managers should construct portfolios along the efficient frontier – that are not dominated in mean-variance by any other.
Individual investors can then choose the portfolio along the efficient frontier that is best suited to their individual levels of risk aversion.