ECON 337901
FINANCIAL ECONOMICS

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The Allais Paradox

As mentioned previously, the independence axiom has been and continues to be a subject of controversy and debate.

Maurice Allais (France, 1911-2010, Nobel Prize 1988) constructed a famous example that illustrates why the independence axiom might not hold in his paper "Le Comportement de L'Homme Rationnel Devant Le Risque: Critique Des Postulats et Axiomes De L'Ecole Americaine," *Econometrica* Vol.21 (October 1953): pp.503-546.

The Allais Paradox

Consider two lotteries:

$$L_1 = \begin{cases} \$10000 & \text{with probability } 0.10 \\ \$0 & \text{with probability } 0.90 \end{cases}$$

$$L_2 = \begin{cases} \$15000 & \text{with probability } 0.09 \\ \$0 & \text{with probability } 0.91 \end{cases}$$

Which would you prefer?

The Allais Paradox

Consider two lotteries:

$$L_1 = \begin{cases} \$10000 & \text{with probability } 0.10 \\ \$0 & \text{with probability } 0.90 \end{cases}$$

$$L_2 = \begin{cases} \$15000 & \text{with probability } 0.09 \\ \$0 & \text{with probability } 0.91 \end{cases}$$

People tend to say $L_2 \succ L_1$.

The Allais Paradox

But now consider other two lotteries:

$$L_3 = \begin{cases} \$10000 & \text{with probability } 1.00 \\ \$0 & \text{with probability } 0.00 \end{cases}$$

$$L_4 = \begin{cases} \$15000 & \text{with probability } 0.90 \\ \$0 & \text{with probability } 0.10 \end{cases}$$

Which would you prefer?

The Allais Paradox

But now consider other two lotteries:

$$L_3 = \begin{cases} \$10000 & \text{with probability } 1.00 \\ \$0 & \text{with probability } 0.00 \end{cases}$$

$$L_4 = \begin{cases} \$15000 & \text{with probability } 0.90 \\ \$0 & \text{with probability } 0.10 \end{cases}$$

Many of the same people who say $L_2 \succ L_1$ often say $L_3 \succ L_4$.

The Allais Paradox

Finally, consider

$$L_5 = \begin{cases} L_3 \text{ (\$10000 for sure)} & \text{with probability } 0.10 \\ \$0 & \text{with probability } 0.90 \end{cases}$$

$$L_6 = \begin{cases} L_4 \text{ (\$15000 w/prob 0.90)} & \text{with probability } 0.10 \\ \$0 & \text{with probability } 0.90 \end{cases}$$

The independence axiom requires anyone who says $L_3 \succ L_4$ to also say that $L_5 \succ L_6$.

The Allais Paradox

But notice that

$$L_5 = \begin{cases} L_3 \text{ (\$10000 for sure)} & \text{with probability } 0.10 \\ \$0 & \text{with probability } 0.90 \end{cases}$$

$$L_6 = \begin{cases} L_4 \text{ (\$15000 w/prob 0.90)} & \text{with probability } 0.10 \\ \$0 & \text{with probability } 0.90 \end{cases}$$

are equivalent to

$$L_5 = \left\{ \begin{array}{ll} \$10000 & \text{with probability } 0.10 \\ \$0 & \text{with probability } 0.90 \end{array} \right\} = L_1$$

$$L_6 = \left\{ \begin{array}{ll} \$15000 & \text{with probability } 0.09 \\ \$0 & \text{with probability } 0.91 \end{array} \right\} = L_2$$

The Allais Paradox

$$L_5 = \begin{cases} L_3 \text{ (\$10000 for sure)} & \text{with probability } 0.10 \\ \$0 & \text{with probability } 0.90 \end{cases}$$

$$L_6 = \begin{cases} L_4 \text{ (\$15000 w/prob 0.90)} & \text{with probability } 0.10 \\ \$0 & \text{with probability } 0.90 \end{cases}$$

are equivalent to

$$L_5 = \left\{ \begin{array}{ll} \$10000 & \text{with probability } 0.10 \\ \$0 & \text{with probability } 0.90 \end{array} \right\} = L_1$$

$$L_6 = \left\{ \begin{array}{ll} \$15000 & \text{with probability } 0.09 \\ \$0 & \text{with probability } 0.91 \end{array} \right\} = L_2$$

The independence axiom requires anyone who says $L_3 \succ L_4$ to also say that $L_5 \succ L_6$. But some of these people say $L_6 = L_2 \succ L_5 = L_1$.

The Allais Paradox

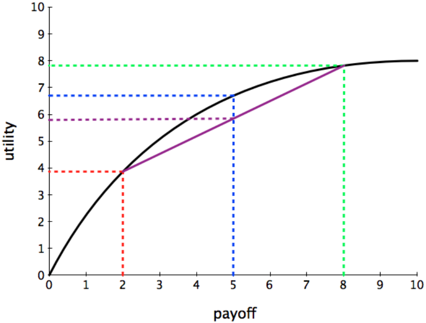
Expected utility remains the dominant framework for analyzing economic decision-making under uncertainty.

But a very active line of ongoing research continues to explore alternatives and generalizations.

Measuring Risk Aversion

We've already seen that within the von Neumann-Morgenstern expected utility framework, risk aversion enters through the concavity of the Bernoulli utility function.

Expected Utility Functions



When u is concave, a payoff of 5 for sure is preferred to a payoff of 8 with probability $1/2$ and 2 with probability $1/2$.

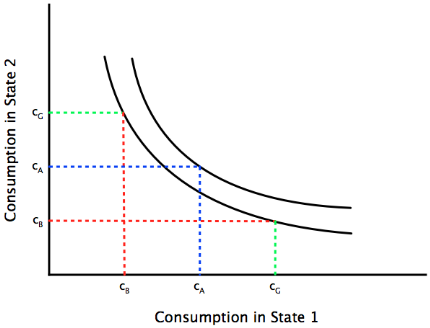
Measuring Risk Aversion

We've also seen previously that concavity of the utility function is related to convexity of indifference curves.

In standard microeconomic theory, this feature of preferences represents a "taste for diversity."

Under uncertainty, it represents a desire to smooth consumption across future states of the world.

Expected Utility Functions



A risk averse consumer prefers $c_A = (c_G + c_B)/2$ in both states to c_G in one state and c_B in the other.

Measuring Risk Aversion

Mathematically, $u'(p) > 0$ means that an investor prefers higher payoffs to lower payoffs, and $u''(p) < 0$ means that the investor is risk averse.

But is there a way of quantifying an investor's degree of risk aversion?

And is there a criterion according to which we might judge one investor to be more risk averse than another?

Measuring Risk Aversion

Since $u''(p) < 0$ makes an investor risk averse, one conjecture would be to say that an investor with Bernoulli utility function $v(p)$ is more risk averse than another investor with Bernoulli utility function $u(p)$ if $v''(p) < u''(p)$ for all payoffs p.

This conjecture works in the example from problem set 9, where investors with "more concave" Bernoulli utility functions display a greater degree of risk aversion.

Measuring Risk Aversion

But does a “more concave” Bernoulli utility function always correspond to greater risk aversion?

Unfortunately, no.

Measuring Risk Aversion

Recall that the preference ordering of an investor with vN-M utility function

$$U(x, y, \pi) = \pi u(x) + (1 - \pi)u(y)$$

is also represented by the vN-M utility function

$$V(x, y, \pi) = \alpha U(x, y, \pi)$$

for any value of $\alpha > 0$.

Measuring Risk Aversion

But with

$$\begin{aligned} V(x, y, \pi) &= \alpha U(x, y, \pi) \\ &= \alpha\pi u(x) + \alpha(1-\pi)u(y) \\ &= \pi\alpha u(x) + (1-\pi)\alpha u(y) \\ &= \pi v(x) + (1-\pi)v(y) \end{aligned}$$

where

$$v(p) = \alpha u(p)$$

for any payoff p.

Measuring Risk Aversion

Now

$$v(p) = \alpha u(p),$$

implies

$$v'(p) = \alpha u'(p)$$

and

$$v''(p) = \alpha u''(p),$$

By making α larger or smaller, the Bernoulli utility function can be made "more" or "less" concave without changing the underlying preference ordering.

Measuring Risk Aversion

In the mid-1960s, Kenneth Arrow and John Pratt proposed two alternative measures of risk aversion that are immune to this problem:

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \text{ coefficient of absolute risk aversion}$$

$$R_R(Y) = -\frac{Yu''(Y)}{u'(Y)} = \text{ coefficient of relative risk aversion}$$

where Y measures the investor's income level.

Measuring Risk Aversion

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \text{ coefficient of absolute risk aversion}$$

$$R_R(Y) = -\frac{Yu''(Y)}{u'(Y)} = \text{ coefficient of relative risk aversion}$$

Absolute risk aversion applies to bets over absolute dollar amounts: $\pm$ \$1000.

Relative risk aversion applies to bets expressed relative to (as a fraction of) income: $\pm$ 1 percent of Y.

Measuring Risk Aversion

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \text{ coefficient of absolute risk aversion}$$

$$R_R(Y) = -\frac{Yu''(Y)}{u'(Y)} = \text{ coefficient of relative risk aversion}$$

Since $v(p) = \alpha u(p)$ implies $v'(p) = \alpha u'(p)$ and $v''(p) = \alpha u''(p)$, these measures are invariant to affine transformations of the Bernoulli utility function.

Measuring Risk Aversion

Two alternative measures of risk aversion are

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \text{ coefficient of absolute risk aversion}$$

$$R_R(Y) = -\frac{Yu''(Y)}{u'(Y)} = \text{ coefficient of relative risk aversion}$$

where Y measures the investor's income level.

And since both measures have a minus sign out in front, both are positive and increase when risk aversion rises.

Measuring Risk Aversion

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \text{ coefficient of absolute risk aversion}$$

$$R_R(Y) = -\frac{Yu''(Y)}{u'(Y)} = \text{ coefficient of relative risk aversion}$$

The notation $R_A(Y)$ and $R_R(Y)$ emphasizes that both measures of risk aversion can depend on the investor's income Y. A given bet can seem more or less risky, depending on the investor's income.

Interpreting the Measures of Risk Aversion

To interpret the two measures of risk aversion, it is helpful to recall from calculus the Taylor approximation of a function f using its derivatives: the first-order approximation

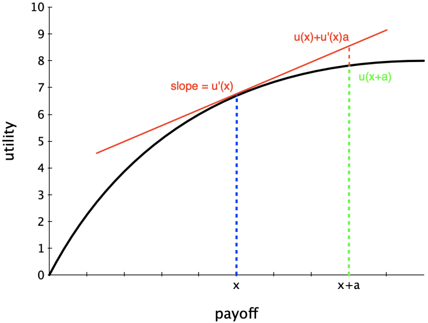
$$f(x+a) \approx f(x) + f'(x)a$$

and the second-order approximation

$$f(x+a) \approx f(x) + f'(x)a + \frac{1}{2}f''(x)a^2.$$

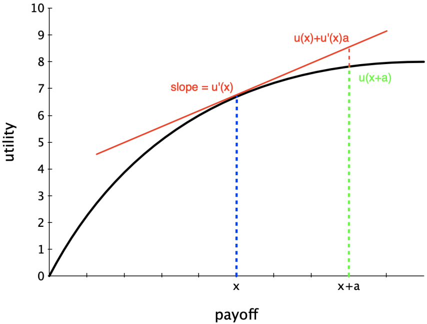
The second-order approximation is more accurate than the first, and both become more accurate as a becomes smaller.

Interpreting the Measures of Risk Aversion



The first-order (linear) approximation $u(x+a) \approx u(x) + u'(x)a$ overstates $u(x+a)$ when u is concave.

Interpreting the Measures of Risk Aversion



Since $u''(x) < 0$, the second-order (quadratic) approximation $u(x+a) \approx u(x) + u'(x)a + (1/2)u''(x)a^2$ will be more accurate.

Interpreting the Measures of Risk Aversion

Focusing first on the measure of absolute risk aversion, consider an investor with initial income Y who is offered a bet: win h with probability π and lose h with probability $1 - \pi$.

A risk-averse investor with vN-M expected utility would never accept this bet if $\pi = 1/2$.

The question is: how much higher than $1/2$ does π have to be to get the investor to accept the bet?

Interpreting the Measures of Risk Aversion

Let π^* be the probability that is just high enough to get the investor to accept the bet.

Then π^* must satisfy

$$u(Y) = \pi^* u(Y+h) + (1-\pi^*) u(Y-h).$$

Interpreting the Measures of Risk Aversion

Take second-order Taylor approximations to $u(Y+h)$ and $u(Y-h)$:

$$u(Y+h) \approx u(Y) + u'(Y)h + \frac{1}{2}u''(Y)h^2$$

$$u(Y-h) \approx u(Y) - u'(Y)h + \frac{1}{2}u''(Y)h^2$$

Interpreting the Measures of Risk Aversion

$$u(Y) = \pi^* u(Y+h) + (1-\pi^*)u(Y-h)$$

$$u(Y+h) \approx u(Y) + u'(Y)h + \frac{1}{2}u''(Y)h^2$$

$$u(Y-h) \approx u(Y) - u'(Y)h + \frac{1}{2}u''(Y)h^2$$

imply

$$\begin{aligned} u(Y) &\approx \pi^* \left[u(Y) + u'(Y)h + \frac{1}{2}u''(Y)h^2 \right] \\ &\quad + (1-\pi^*) \left[u(Y) - u'(Y)h + \frac{1}{2}u''(Y)h^2 \right] \end{aligned}$$

Interpreting the Measures of Risk Aversion

$$
\begin{aligned}
u(Y) \;\approx\;& \pi^* \left[u(Y) + u'(Y)h + \frac{1}{2}u''(Y)h^2 \right] \\
&+ (1-\pi^*) \left[u(Y) - u'(Y)h + \frac{1}{2}u''(Y)h^2 \right]
\end{aligned}
$$

implies

$$
u(Y) \approx u(Y) + (2\pi^* - 1)u'(Y)h + \frac{1}{2}u''(Y)h^2
$$

Interpreting the Measures of Risk Aversion

$$u(Y) \approx u(Y) + (2\pi^* - 1)u'(Y)h + \frac{1}{2}u''(Y)h^2$$

$$0 \approx (2\pi^* - 1)u'(Y)h + \frac{1}{2}u''(Y)h^2$$

$$0 \approx (2\pi^* - 1)u'(Y) + \frac{1}{2}u''(Y)h$$

$$2\pi^* u'(Y) \approx u'(Y) - \frac{1}{2}u''(Y)h$$

$$\pi^* \approx \frac{1}{2} + \frac{1}{4}\left[-\frac{u''(Y)}{u'(Y)}\right] h$$

Interpreting the Measures of Risk Aversion

Since

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \text{ coefficient of absolute risk aversion,}$$

it follows from these calculations that

$$\pi^* \approx \frac{1}{2} + \frac{1}{4}\left[-\frac{u''(Y)}{u'(Y)}\right]h = \frac{1}{2} + \frac{1}{4}hR_A(Y) > \frac{1}{2}.$$

The boost in π above $1/2$ required for an investor with income Y to accept a bet of plus or minus h relates directly to the coefficient of absolute risk aversion.

Interpreting the Measures of Risk Aversion

As an example, suppose that we ask an investor: What value of π^* would you need to accept a bet of plus-or-minus $h = \$1000$?

And the investor says: I'll take it if $\pi^* = 0.75$.

Interpreting the Measures of Risk Aversion

With $h = \$1000$ and $\pi^* = 0.75$,

$$\pi^* \approx \frac{1}{2} + \frac{1}{4} h R_A(Y)$$

implies

$$0.75 \approx 0.50 + \frac{1000}{4} R_A(Y)$$

$$0.25 \approx 250 R_A(Y)$$

$$R_A(Y) \approx \frac{0.25}{250} = 0.001.$$

Interpreting the Measures of Risk Aversion

Realistically, a bet over \$1000 is probably going to seem more risky to someone who starts out with less income.

In general, $R_A(Y)$ can depend on Y. More specifically, it seems likely that $R_A(Y)$ decreases when Y goes up, so that

$$R_A'(Y) < 0.$$

Interpreting the Measures of Risk Aversion

Absolute risk aversion describes an investor's attitude towards absolute bets of plus or minus h.

A similar analysis shows that relative risk aversion describes an investor's attitude towards relative bets of plus or minus kY, so that now, k is a fraction of total income.

Interpreting the Measures of Risk Aversion

Consider an investor with initial income Y who is offered a bet: win kY with probability π and lose kY with probability $1-\pi$.

A risk-averse investor with vN-M expected utility would never accept this bet if $\pi = 1/2$.

The question is: how much higher than $1/2$ does π have to be to get the investor to accept the bet?

Interpreting the Measures of Risk Aversion

Let π^* be the probability that is just high enough to get the investor to accept the bet.

Now π^* must satisfy

$$u(Y) = \pi^* u(Y + Yk) + (1 - \pi^*) u(Y - Yk).$$

Interpreting the Measures of Risk Aversion

Take second-order Taylor approximations to $u(Y + Yk)$ and $u(Y - Yk)$:

$$u(Y + Yk) \approx u(Y) + u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2$$

$$u(Y - Yk) \approx u(Y) - u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2$$

Interpreting the Measures of Risk Aversion

$$u(Y) = \pi^* u(Y + Yk) + (1 - \pi^*) u(Y - Yk)$$

$$u(Y + Yk) \approx u(Y) + u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2$$

$$u(Y - Yk) \approx u(Y) - u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2$$

imply

$$\begin{aligned} u(Y) \approx\ & \pi^* \left[u(Y) + u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2\right] \\ & + (1 - \pi^*) \left[u(Y) - u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2\right] \end{aligned}$$

Interpreting the Measures of Risk Aversion

$$
\begin{aligned}
u(Y) \approx\ & \pi^* \left[u(Y) + u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2 \right] \\
& + (1-\pi^*) \left[u(Y) - u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2 \right]
\end{aligned}
$$

implies

$$
u(Y) \approx u(Y) + (2\pi^* - 1)u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2
$$

Interpreting the Measures of Risk Aversion

$$u(Y) \approx u(Y) + (2\pi^* - 1)u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2$$

$$0 \approx (2\pi^* - 1)u'(Y)Yk + \frac{1}{2}u''(Y)(Yk)^2$$

$$0 \approx (2\pi^* - 1)u'(Y) + \frac{1}{2}u''(Y)Yk$$

$$2\pi^* u'(Y) \approx u'(Y) - \frac{1}{2}u''(Y)Yk$$

$$\pi^* \approx \frac{1}{2} + \frac{1}{4}\left[-\frac{Yu''(Y)}{u'(Y)}\right]k$$

Interpreting the Measures of Risk Aversion

Since

$$R_R(Y) = -\frac{Yu''(Y)}{u'(Y)} = \text{ coefficient of relative risk aversion,}$$

it follows from these calculations that

$$\pi^* \approx \frac{1}{2} + \frac{1}{4}\left[-\frac{Yu''(Y)}{u'(Y)}\right]k = \frac{1}{2} + \frac{1}{4}kR_R(Y) > \frac{1}{2}.$$

The boost in π above $1/2$ required for an investor with income Y to accept a bet of plus or minus kY relates directly to the coefficient of relative risk aversion.

Interpreting the Measures of Risk Aversion

Suppose that we ask an investor: What value of π^* would you need to accept a bet of plus-or-minus one percent ($k = 0.01$) of your income?

And the investor says: I'll take it if $\pi^* = 0.75$.

Interpreting the Measures of Risk Aversion

With $k = 0.01$ and $\pi^* = 0.75$,

$$\pi^* \approx \frac{1}{2} + \frac{1}{4} k R_R(Y)$$

implies

$$0.75 \approx 0.50 + \frac{0.01}{4} R_R(Y)$$

$$0.25 \approx 0.0025 R_R(Y)$$

$$R_R(Y) \approx \frac{0.25}{0.0025} = 100.$$

Interpreting the Measures of Risk Aversion

Again, our notation $R_R(Y)$ allows relative risk aversion to depend on income Y.

On the other hand, since the coefficient of relative risk aversion describes aversion to risk over bets that are expressed relative to income, it is more plausible to assume that investors have constant relative risk aversion.

Interpreting the Measures of Risk Aversion

Suppose, therefore, that the Bernoulli utility function takes the form

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1-\gamma}$$

where $\gamma > 0$. For this function, Guillaume de l'Hôpital's (France, 1661-1704) rule implies that when $\gamma = 1$

$$\frac{Y^{1-\gamma} - 1}{1-\gamma} = \ln(Y),$$

where ln denotes the natural logarithm. This was the form that Daniel Bernoulli used to describe preferences over payoffs.

Interpreting the Measures of Risk Aversion

With

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma}$$

it follows that

$$u'(Y) = Y^{-\gamma}$$

$$u''(Y) = -\gamma Y^{-\gamma - 1}$$

$$R_R(Y) = -\frac{Yu''(Y)}{u'(Y)} = \frac{Y \gamma Y^{-\gamma-1}}{Y^{-\gamma}} = \gamma,$$

so that this utility function displays constant relative risk aversion, which does not depend on income.

Interpreting the Measures of Risk Aversion

So if we were willing to make the assumption of constant relative risk aversion, we could use the results from our example, where an investor requires $\pi^* = 0.75$ to accept a bet with $k = 0.01$ to set $\gamma = 100$ in

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma}$$

and thereby tailor portfolio decisions specifically for this investor.