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FINANCIAL ECONOMICS

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3 Making Choices in Risky Situations

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Criteria for Choice Over Risky Prospects

In the broadest sense, “risk” refers to uncertainty about the future cash flows provided by a financial asset.

A more specific way of modeling risk is to think of those cash flows as varying across different states of the world in future periods . . .

. . . that is, to describe future cash flows as random variables.

Criteria for Choice Over Risky Prospects

Consider three assets:

		Payoffs Next Year in	
	Price Today	Good State	Bad State
Asset 1	-1000	1200	1050
Asset 2	-1000	1600	500
Asset 3	-1000	1600	1050

where the good and bad states occur with equal probability ($\pi = 1 - \pi = 1/2$).

Criteria for Choice Over Risky Prospects

		Payoffs Next Year in	
	Price Today	Good State	Bad State
Asset 1	-1000	1200	1050
Asset 2	-1000	1600	500
Asset 3	-1000	1600	1050

Asset 3 exhibits state-by-state dominance over assets 1 and 2. Any investor who prefers more to less would always choose asset 3 above the others.

Criteria for Choice Over Risky Prospects

In general, one asset displays state-by-state dominance over another if:

1. It pays off at least as much in all states

AND

2. It pays off more in at least one state,

so investors who prefer more to less will never regret buying it.

Criteria for Choice Over Risky Prospects

		Payoffs Next Year in	
	Price Today	Good State	Bad State
Asset 1	-1000	1200	1050
Asset 2	-1000	1600	500
Asset 3	-1000	1600	1050

Although asset 3 is unambiguously the best, the choice between assets 1 and 2 seems less clear cut.

Criteria for Choice Over Risky Prospects

It can often be helpful to convert prices and payoffs to percentage returns:

		Payoffs Next Year in	
	Price Today	Good State	Bad State
Asset 1	-1000	1200	1050
Asset 2	-1000	1600	500
Asset 3	-1000	1600	1050

	Percentage Return in	
	Good State	Bad State
Asset 1	20	5
Asset 2	60	-50
Asset 3	60	5

Criteria for Choice Over Risky Prospects

In probability theory, if a random variable $\tilde{X}$ can take on n possible values, $X_1, X_2, \ldots, X_n$, with probabilities $\pi_1, \pi_2, \ldots, \pi_n$, then the expected value of $\tilde{X}$ is

$$E(\tilde{X}) = \pi_1 X_1 + \pi_2 X_2 + \ldots + \pi_n X_n,$$

the variance (or mean squared deviation) of $\tilde{X}$ is

$$\begin{aligned}\sigma^2(\tilde{X}) &= \pi_1[X_1 - E(\tilde{X})]^2 \\ &\quad + \pi_2[X_2 - E(\tilde{X})]^2 + \ldots + \pi_n[X_n - E(\tilde{X})]^2,\end{aligned}$$

and the standard deviation of $\tilde{X}$ is $\sigma(\tilde{X}) = [\sigma^2(\tilde{X})]^{1/2}$.

Criteria for Choice Over Risky Prospects

	Percentage Return in	
	Good State	Bad State
Asset 1	20	5
Asset 2	60	-50
Asset 3	60	5

$$E(R_1) = (1/2)20 + (1/2)5 = 12.5$$

$$\sigma(R_1) = [(1/2)(20 - 12.5)^2 + (1/2)(5 - 12.5)^2]^{1/2} = 7.5$$

Criteria for Choice Over Risky Prospects

	Percentage Return in			
	Good State	Bad State	$E(R)$	$\sigma(R)$
Asset 1	20	5	12.5	7.5
Asset 2	60	-50		
Asset 3	60	5		

$$E(R_2) = (1/2)60 + (1/2)(-50) = 5$$

$$\sigma(R_2) = [(1/2)(60-5)^2 + (1/2)(-50-5)^2]^{1/2} = 55$$

Criteria for Choice Over Risky Prospects

	Percentage Return in			
	Good State	Bad State	$E(R)$	$\sigma(R)$
Asset 1	20	5	12.5	7.5
Asset 2	60	-50	5	55
Asset 3	60	5		

$$E(R_3) = (1/2)60 + (1/2)5 = 32.5$$

$$\sigma(R_3) = [(1/2)(60 - 32.5)^2 + (1/2)(5 - 32.5)^2]^{1/2} = 27.5$$

Criteria for Choice Over Risky Prospects

	Percentage Return in			
	Good State	Bad State	$E(R)$	$\sigma(R)$
Asset 1	20	5	12.5	7.5
Asset 2	60	-50	5	55
Asset 3	60	5	32.5	27.5

Asset 1 exhibits mean-variance dominance over asset 2, since it offers a higher expected return with lower variance.

Criteria for Choice Over Risky Prospects

In general, one asset displays mean-variance dominance over another if:

1. $E(R_1) > E(R_2)$ and $\sigma(R_1) \leq \sigma(R_2)$

so that it offers a higher expected return with no greater standard deviation,

OR

2. $E(R_1) \geq E(R_2)$ and $\sigma(R_1) < \sigma(R_2)$

so that it offers a smaller standard deviation and no less expected return.

Criteria for Choice Over Risky Prospects

	Percentage Return in			
	Good State	Bad State	$E(R)$	$\sigma(R)$
Asset 1	20	5	12.5	7.5
Asset 2	60	-50	5	55
Asset 3	60	5	32.5	27.5

But notice that by the mean-variance criterion, asset 3 dominates asset 2 but not asset 1, even though on a state-by-state basis, asset 3 is clearly to be preferred.

Criteria for Choice Over Risky Prospects

Consider two more assets:

	Percentage Return in			
	Good State	Bad State	$E(R)$	$\sigma(R)$
Asset 4	5	3		
Asset 5	8	2		

Again, neither exhibits state-by-state dominance, so let's try to use the mean-variance criterion again.

Criteria for Choice Over Risky Prospects

	Percentage Return in			
	Good State	Bad State	$E(R)$	$\sigma(R)$
Asset 4	5	3		
Asset 5	8	2		

$$E(R_4) = (1/2)5 + (1/2)3 = 4$$

$$\sigma(R_4) = [(1/2)(5-4)^2 + (1/2)(3-4)^2]^{1/2} = 1$$

$$E(R_5) = (1/2)8 + (1/2)2 = 5$$

$$\sigma(R_5) = [(1/2)(8-5)^2 + (1/2)(2-5)^2]^{1/2} = 3$$

Criteria for Choice Over Risky Prospects

	Percentage Return in			
	Good State	Bad State	$E(R)$	$\sigma(R)$
Asset 4	5	3	4	1
Asset 5	8	2	5	3

Neither asset exhibits mean-variance dominance either.

Criteria for Choice Over Risky Prospects

	Percentage Return in			
	Good State	Bad State	$E(R)$	$\sigma(R)$
Asset 4	5	3	4	1
Asset 5	8	2	5	3

William Sharpe (US, b.1934, Nobel Prize 1990) suggested that in these circumstances, it can help to compare the two assets' Sharpe ratios, defined as $E(R)/\sigma(R)$.

Criteria for Choice Over Risky Prospects

Note: in practice, the Sharpe ratio is usually defined as the expected "excess return" above the risk-free rate r_f divided by the standard deviation:

$$\frac{E(R) - r_f}{\sigma(R)}.$$

For these preliminary examples, we are either assuming that $r_f = 0$ or using $E(R)/\sigma(R)$ as a simplified definition of the Sharpe ratio.

Criteria for Choice Over Risky Prospects

	Percentage Return in				
	Good State	Bad State	$E(R)$	$\sigma(R)$	E/σ
Asset 4	5	3	4	1	4
Asset 5	8	2	5	3	1.67

Comparing Sharpe ratios, asset 4 is preferred to asset 5.

Criteria for Choice Over Risky Prospects

	Percentage Return in				
	Good State	Bad State	$E(R)$	$\sigma(R)$	E/σ
Asset 4	5	3	4	1	4
Asset 5	8	2	5	3	1.67

But using the Sharpe ratio to choose between assets means assuming that investors "weight" the mean and standard deviation equally, in the sense that a doubling of $\sigma(R)$ is adequately compensated by a doubling of $E(R)$. Investors who are more or less averse to risk will disagree.

Criteria for Choice Over Risky Prospects

1. State-by-state dominance is the most robust criterion, but often cannot be applied.
2. Mean-variance dominance is more widely-applicable, but can sometimes be misleading and cannot always be applied.
3. The Sharpe ratio can always be applied, but requires a very specific assumption about consumer attitudes towards risk.

We need a more careful and comprehensive approach to comparing random cash flows.

Preferences and Utility Functions

Of course, economists face a more general problem of this kind.

Even if we accept that more (of everything) is preferred to less, how do consumers compare different "bundles" of goods that may contain more of one good but less of another?

Microeconomists have identified a set of conditions that allow a consumer's preferences to be described by a utility function.

Preferences and Utility Functions

Gerard Debreu identified these conditions in 1954:

Cowles Foundation Paper 97
Reprinted from Thrall, Davis and Coombs , eds.,
Decision Processes, John Wiley, 1954

CHAPTER XI

REPRESENTATION OF A PREFERENCE ORDERING BY A NUMERICAL FUNCTION*

by

Gerard Debreu

COWLES COMMISSION FOR RESEARCH IN ECONOMICS

Preferences and Utility Functions

Let a, b, and c represent three bundles of goods.

These may be arbitrarily long lists, or vectors ($a \in \mathbb{R}^N$), indicating how much of each of an arbitrarily large number of goods is included in the bundle.

A preference relation $\succeq$ can be used to represent the consumer's preferences over different consumption bundles.

Preferences and Utility Functions

The expression

$$a \succ b$$

indicates that the consumer strictly prefers a to b,

$$a \sim b$$

indicates that the consumer is indifferent between a and b, and

$$a \succeq b$$

indicates that the consumer either strictly prefers or is indifferent between a and b.

Preferences and Utility Functions

A1 The preference relation is assumed to be complete: For any two bundles a and b, either $a \succeq b$, $b \succeq a$, or both, and in the latter case $a \sim b$.

The consumer has to decide whether he or she prefers one bundle to another or is indifferent between the two. Ambiguous tastes are not allowed.

Preferences and Utility Functions

A2 The preference relation is assumed to be transitive: For any three bundles a, b, and c, if $a \succeq b$ and $b \succeq c$, then $a \succeq c$.

The consumer's tastes must be consistent in this sense. Together, (A1) and (A2) require the consumer to be fully informed and rational.

Preferences and Utility Functions

A3 The preference relation is assumed to be continuous: If $\{a_n\}$ and $\{b_n\}$ are two sequences of bundles such that $a_n \to a$, $b_n \to b$, and $a_n \succeq b_n$ for all n, then $a \succeq b$.

Very small changes in consumption bundles cannot lead to large changes in preferences over those bundles.

Preferences and Utility Functions

An two-good example that violates (A3) is the case of lexicographic preferences:

$$a = (a_1, a_2) \succ b = (b_1, b_2) \quad \text{if} \quad a_1 > b_1$$
$$\text{or} \quad a_1 = b_1 \text{ and } a_2 > b_2.$$

This example shows that "continuity" of preferences requires that consumers be willing to accept trade offs.

Preferences and Utility Functions

The following theorem was proven by Gerard Debreu in 1954.

Theorem If preferences are complete, transitive, and continuous, then they can be represented by a continuous, real-valued utility function. That is, if (A1)-(A3) hold, there is a continuous function $u : \mathbb{R}^n \mapsto \mathbb{R}$ such that for any two consumption bundles a and b,

$$a \succeq b \text{ if and only if } u(a) \geq u(b).$$

Preferences and Utility Functions

Note that if preferences are represented by the utility function u,

$$a \succeq b \text{ if and only if } u(a) \geq u(b),$$

then they are also represented by the utility function v, where

$$v(a) = F(u(a))$$

and $F : \mathbb{R} \mapsto \mathbb{R}$ is any increasing function.

The concept of utility as it is used in standard microeconomic theory is ordinal, as opposed to cardinal.

Preferences and Utility Functions

As an example illustrating that utility is ordinal, not cardinal, suppose that a consumer has preferences that are described by the utility function

$$c_a^{\alpha} c_b^{1-\alpha}$$

The consumer's problem is then

$$\max_{c_a, c_b} \; c_a^{\alpha} c_b^{1-\alpha} \text{ subject to } Y \geq p_a c_a + p_b c_b$$

Preferences and Utility Functions

The consumer's problem is then

$$\max_{c_a, c_b} \; c_a^{\alpha} c_b^{1-\alpha} \text{ subject to } Y \geq p_a c_a + p_b c_b$$

Set up the Lagrangian

$$L(c_a, c_b, \lambda) = c_a^{\alpha} c_b^{1-\alpha} + \lambda(Y - p_a c_a - p_b c_b)$$

Preferences and Utility Functions

Set up the Lagrangian

$$L(c_a, c_b, \lambda) = c_a^{\alpha} c_b^{1-\alpha} + \lambda(Y - p_a c_a - p_b c_b)$$

Take the first-order conditions:

$$\alpha c_a^{*\alpha-1} c_b^{*1-\alpha} - \lambda^* p_a = 0$$

$$(1-\alpha) c_a^{*\alpha} c_b^{*-\alpha} - \lambda^* p_b = 0$$

Preferences and Utility Functions

The first-order conditions

$$\alpha c_a^{*\alpha-1} c_b^{*1-\alpha} - \lambda^* p_a = 0$$

$$(1-\alpha) c_a^{*\alpha} c_b^{*-\alpha} - \lambda^* p_b = 0$$

and the binding constraint

$$Y = p_a c_a^* + p_b c_b^*$$

form a system of 3 equations in 3 unknowns: c_a^*, c_b^*, and λ^*.

Preferences and Utility Functions

The first-order conditions

$$\alpha c_a^{*\alpha-1} c_b^{*1-\alpha} - \lambda^* p_a = 0$$

$$(1-\alpha) c_a^{*\alpha} c_b^{*-\alpha} - \lambda^* p_b = 0$$

Imply that

$$\frac{\alpha c_a^{*\alpha-1} c_b^{*1-\alpha}}{(1-\alpha) c_a^{*\alpha} c_b^{*-\alpha}} = \frac{\lambda^* p_a}{\lambda^* p_b}$$

Preferences and Utility Functions

The first-order conditions imply that

$$\frac{\alpha c_a^{*\alpha-1} c_b^{*1-\alpha}}{(1-\alpha) c_a^{*\alpha} c_b^{*-\alpha}} = \frac{\lambda^* p_a}{\lambda^* p_b}$$

$$\frac{\alpha c_b^*}{(1-\alpha) c_a^*} = \frac{p_a}{p_b}$$

$$c_b^* = \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{p_a}{p_b}\right) c_a^*.$$

Preferences and Utility Functions

Substitute

$$c_b^* = \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{p_a}{p_b}\right)c_a^*.$$

into the budget constraint to find

$$\begin{aligned} Y &= p_a c_a^* + p_b c_b^* = p_a c_a^* + p_b \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{p_a}{p_b}\right)c_a^* \\ &= p_a c_a^* + \left(\frac{1}{\alpha} - 1\right) p_a c_a^* = \frac{p_a c_a^*}{\alpha} \end{aligned}$$

or

$$c_a^* = \frac{\alpha Y}{p_a}$$

Preferences and Utility Functions

Then substitute

$$c_a^* = \frac{\alpha Y}{p_a}$$

back into

$$c_b^* = \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{p_a}{p_b}\right)c_a^*.$$

to find

$$c_b^* = \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{p_a}{p_b}\right)\frac{\alpha Y}{p_a} = \frac{(1-\alpha)Y}{p_b}$$

The same as in problem set 2!

Preferences and Utility Functions

The solutions

$$c_a^* = \frac{\alpha Y}{p_a} \text{ and } c_b^* = \frac{(1-\alpha)Y}{p_b}$$

are the same as in problem set 2. Why?

Take the natural logarithm of the utility function

$$\ln(c_a^\alpha c_b^{1-\alpha}) = \ln(c_a^\alpha) + \ln(c_b^{1-\alpha}) = \alpha \ln(c_a) + (1-\alpha)\ln(c_b)$$

These two utility functions describe the *same* preference ordering.