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FINANCIAL ECONOMICS

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Using Options to Infer Contingent Claims Prices

Notice that call options often have payoff structures that resemble those of contingent claims: making positive payoffs in “good” states and expiring worthless in “bad” states.

Douglas Breeden (US, b.1950) and Robert Litzenberger (US, b.1943) devised a way of using option prices to infer contingent claims prices, allowing for many states of the world.

Douglas Breeden and Robert Litzenberger, “Prices of State-Contingent Claims Implicit in Option Prices,” *Journal of Business* Vol.51 (October 1978): pp.621-651.

Using Options to Infer Contingent Claims Prices

Suppose there are N states, corresponding to different levels of the S&P 500, with

$$P^1 < P^2 < \ldots < P^N$$

and

$$P^{i+1} = P^i + \delta$$

with $\delta > 0$.

Using Options to Infer Contingent Claims Prices

For each state i, construct a "butterfly" portfolio of call options:

Buy one calls with strike price P^{i-1}

Write (sell short) two calls with strike price P^i

Buy one call with strike price P^{i+1}

If q_o^i is the price of an option with strike price P^i, this portfolio costs

$$q_o^{i-1} + q_o^{i+1} - 2q_o^i$$

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Now let's compute the portfolio's payoffs:

S&P 500	Long $K = P^{i-1}$	Short 2 $K = P^{i}$	Long $K = P^{i+1}$	Total
$P \leq P^{i-1}$	0	0	0	0
$P = P^{i}$	$P^{i} - P^{i-1}$	0	0	δ
$P \geq P^{i+1}$	$P - P^{i-1}$	$-2(P - P^{i})$	$P - P^{i+1}$	0

$$
\begin{aligned}
& (P - P^{i-1}) - 2(P - P^{i}) + (P - P^{i+1}) \\
= \; & 2P^{i} - P^{i-1} - P^{i+1} \\
= \; & 2P^{i} - (P^{i} - \delta) - (P^{i} + \delta) = 0
\end{aligned}
$$

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$P \leq P^{i-1}$	0	0	0	0
$P = P^{i}$	$P^{i} - P^{i-1}$	0	0	δ
$P \geq P^{i+1}$	$P - P^{i-1}$	$-2(P - P^{i})$	$P - P^{i+1}$	0

Since the portfolio's payoffs replicate those from δ claims for state i, the price q^{i}_{cc} of a contingent claim for state i must satisfy

$$q^{i}_{cc} = (1/\delta)(q^{i-1}_{o} + q^{i+1}_{o} - 2q^{i}_{o})$$

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$$q_{cc}^{i} = (1/\delta)(q_o^{i-1} + q_o^{i+1} - 2q_o^{i})$$

Additional accuracy can be achieved by choosing smaller values of δ, that is, by using a "finer grid" to define the states.

Options trade on the S&P 500 with many strike prices, so data on q_o^i are readily available.

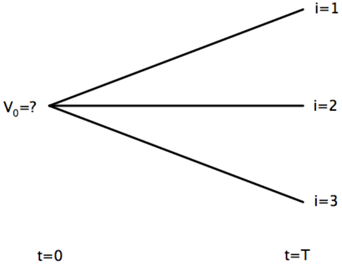
Note that you don't have to actually trade the options to price the contingent claims.

Black-Scholes Option Pricing

Black and Scholes and Merton considered a more general setting, in which the option priced at $t = 0$ does not expire until $t = T$.

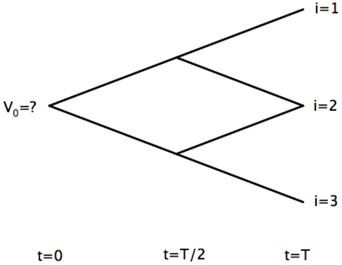
They also allowed for (many) more than two possible states at $t = T$.

Black-Scholes Option Pricing



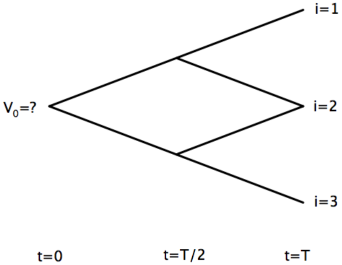
The technical problem is that with more than two states at $t = T$, more than two assets are needed to create a portfolio with the same payoffs as the option.

Black-Scholes Option Pricing



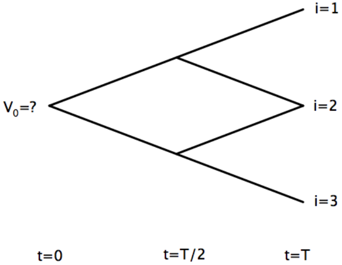
Black and Scholes and Merton realized that this problem can be solved by breaking the full period into sub-periods, so that there are only two states in each sub-period.

Black-Scholes Option Pricing



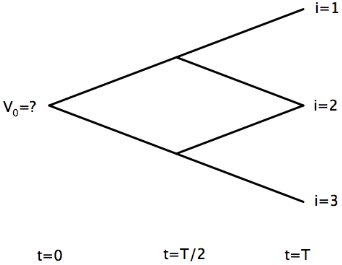
With three states at $t = T$, only two subperiods are needed, but with many states at $t = T$, many subperiods are needed.

Black-Scholes Option Pricing



A dynamic hedging strategy can then be used to track the payoffs on the option using a portfolio consisting only of the stock and bond ...

Black-Scholes Option Pricing



. . . but where the number of shares and the number of bonds must be adjusted in each subperiod so that the portfolio can continue to track the option's payoffs.

Black-Scholes Option Pricing

Black and Scholes used methods in stochastic calculus developed by Kiyoshi Ito (Japan, 1915-2008) in the 1940s and early 1950s to show that in the more general case, the solution for the option price is

$$q^0 = N_1 q^s - N_2 q^b K = N_1 q^s - N_2 \left(\frac{K}{1 + r_f} \right)$$

where $N_1 = F(d_1)$ and $N_2 = F(d_2)$,

$$d_1 = \frac{\ln(q^s/K) + (r_f + \sigma^2/2)T}{\sigma\sqrt{T}} \text{ and } d_2 = d_1 - \sigma\sqrt{T}$$

σ is the standard deviation of the return on the stock, and F is the standard normal cumulative distribution function, so that $F(X)$ measures the probability that a random variable that is normally distributed with mean zero and variance one turns out to be less than or equal to X.

Black-Scholes Option Pricing

To use the Black-Scholes formula

$$q^0 = N_1 q^s - N_2 q^b K = N_1 q^s - N_2 \left(\frac{K}{1 + r_f} \right)$$

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$$d_1 = \frac{\ln(q^s/K) + (r_f + \sigma^2/2)T}{\sigma\sqrt{T}} \text{ and } d_2 = d_1 - \sigma\sqrt{T}$$

you need an estimate of σ, the standard deviation of the return on the stock.

Black-Scholes Option Pricing

Alternatively, if you see the price of a traded option, you can use the Black-Scholes formula

$$q^0 = N_1 q^s - N_2 q^b K = N_1 q^s - N_2 \left(\frac{K}{1 + r_f} \right)$$

where $N_1 = F(d_1)$ and $N_2 = F(d_2)$,

$$d_1 = \frac{\ln(q^s/K) + (r_f + \sigma^2/2)T}{\sigma\sqrt{T}} \text{ and } d_2 = d_1 - \sigma\sqrt{T}$$

to estimate σ, the standard deviation of the return on the stock. In fact, the VIX volatility index is similar to the σ implied by the price of an option on the S&P 500.