Using Options to Infer Contingent Claims Prices

Notice that call options often have payoff structures that resemble those of contingent claims: making positive payoffs in “good” states and expiring worthless in “bad” states.

Douglas Breeden (US, b.1950) and Robert Litzenberger (US, b.1943) devised a way of using option prices to infer contingent claims prices, allowing for many states of the world.

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Suppose there are $N$ states, corresponding to different levels of the S&P 500, with

$$P^1 < P^2 < \ldots < P^N$$

and

$$P^{i+1} = P^i + \delta$$

with $\delta > 0$. 
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For each state \( i \), construct a “butterfly” portfolio of call options:

- Buy one calls with strike price \( P_{i-1} \)
- Write (sell short) two calls with strike price \( P_i \)
- Buy one call with strike price \( P_{i+1} \)

If \( q^i_o \) is the price of an option with strike price \( P^i \), this portfolio costs

\[
q^{i-1}_o + q^{i+1}_o - 2q^i_o
\]
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Now let’s compute the portfolio’s payoffs:

<table>
<thead>
<tr>
<th>S&amp;P 500</th>
<th>Long</th>
<th>Short 2</th>
<th>Long</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>K = $P^{i-1}$</td>
<td>$0$</td>
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<td>$0$</td>
</tr>
<tr>
<td>$P = P^i$</td>
<td>$P^i - P^{i-1}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$\delta$</td>
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<td>$P - P^{i-1}$</td>
<td>$-2(P - P^i)$</td>
<td>$P - P^{i+1}$</td>
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\[
(P - P^{i-1}) - 2(P - P^i) + (P - P^{i+1})
= 2P^i - P^{i-1} - P^{i+1}
= 2P^i - (P^i - \delta) - (P^i + \delta) = 0
\]
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Since the portfolio’s payoffs replicate those from $\delta$ claims for state $i$, the price $q_{cc}^i$ of a contingent claim for state $i$ must satisfy

$$q_{cc}^i = \frac{1}{\delta}(q_{o}^{i-1} + q_{o}^{i+1} - 2q_{o}^i)$$
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\[ q_{cc}^i = \frac{1}{\delta}(q_{o}^{i-1} + q_{o}^{i+1} - 2q_{o}^i) \]

Additional accuracy can be achieved by choosing smaller values of \( \delta \), that is, by using a “finer grid” to define the states.

Options trade on the S&P 500 with many strike prices, so data on \( q_{o}^i \) are readily available.

Note that you don’t have to actually trade the options to price the contingent claims.
Black and Scholes and Merton considered a more general setting, in which the option priced at $t = 0$ does not expire until $t = T$.

They also allowed for (many) more than two possible states at $t = T$. 
Black-Scholes Option Pricing

The technical problem is that with more than two states at $t = T$, more than two assets are needed to create a portfolio with the same payoffs as the option.
Black and Scholes and Merton realized that this problem can be solved by breaking the full period into sub-periods, so that there are only two states in each sub-period.
Black-Scholes Option Pricing

With three states at $t = T$, only two subperiods are needed, but with many states at $t = T$, many subperiods are needed.
A dynamic hedging strategy can then be used to track the payoffs on the option using a portfolio consisting only of the stock and bond . . .
Black-Scholes Option Pricing

... but where the number of shares and the number of bonds must be adjusted in each subperiod so that the portfolio can continue to track the option’s payoffs.
Black-Scholes Option Pricing

Black and Scholes used methods in stochastic calculus developed by Kiyoshi Ito (Japan, 1915-2008) in the 1940s and early 1950s to show that in the more general case, the solution for the option price is

\[ q^0 = N_1 q^s - N_2 q^b K = N_1 q^s - N_2 \left( \frac{K}{1 + r_f} \right) \]

where \( N_1 = F(d_1) \) and \( N_2 = F(d_2) \),

\[ d_1 = \frac{\ln(q^s/K) + (r_f + \sigma^2/2) T}{\sigma \sqrt{T}} \]

and \( d_2 = d_1 - \sigma \sqrt{T} \)

\( \sigma \) is the standard deviation of the return on the stock, and \( F \) is the standard normal cumulative distribution function, so that \( F(X) \) measures the probability that a random variable that is normally distributed with mean zero and variance one turns out to be less than or equal to \( X \).
Black-Scholes Option Pricing

To use the Black-Scholes formula

\[ q^0 = N_1 q^s - N_2 q^b K = N_1 q^s - N_2 \left( \frac{K}{1 + r_f} \right) \]

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\[ d_1 = \frac{\ln(q^s/K) + (r_f + \sigma^2/2) T}{\sigma \sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma \sqrt{T} \]

you need an estimate of \( \sigma \), the standard deviation of the return on the stock.
Black-Scholes Option Pricing

Alternatively, if you see the price of a traded option, you can use the Black-Scholes formula

\[
q^0 = N_1 q^s - N_2 q^b K = N_1 q^s - N_2 \left( \frac{K}{1 + r_f} \right)
\]

where \(N_1 = F(d_1)\) and \(N_2 = F(d_2)\),

\[
d_1 = \frac{\ln(q^s/K) + (r_f + \sigma^2/2)T}{\sigma \sqrt{T}}\quad \text{and} \quad d_2 = d_1 - \sigma \sqrt{T}
\]

to estimate \(\sigma\), the standard deviation of the return on the stock. In fact, the VIX volatility index is similar to the \(\sigma\) implied by the price of an option on the S&P 500.