ECON 337901
FINANCIAL ECONOMICS

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February 25, 2020

Consumer Optimization: The Risk Dimension

Consider buying s shares of stock and b bonds, in order to replicate the contingent claim for the bad state.

In the good state, the payoffs should be

$$sd^G + b = 0$$

and in the bad state, the payoffs should be

$$sd^B + b = 1$$

since the contingent claim pays off one in the bad state and zero in the good state.

Consumer Optimization: The Risk Dimension

To replicate the contingent claim for the bad state:

$$sd^G + b = 0 \Rightarrow b = -sd^G$$
$$sd^B + b = 1$$

Substitute the first equation into the second to solve for

$$s = \frac{-1}{d^G - d^B} \text{ and } b = \frac{d^G}{d^G - d^B}$$

Once again, this requires going long one asset and short the other.

Consumer Optimization: The Risk Dimension

To replicate the contingent claim for the bad state:

$$s = \frac{-1}{d^G - d^B} \text{ and } b = \frac{d^G}{d^G - d^B}$$

Once again, if we know the prices q^{stock} and q^{bond} of the stock and bond, we can infer that in the absence of arbitrage, the claim for the bad state would have price

$$q^B = q^{stock} s + q^{bond} b = \frac{d^G q^{bond} - q^{stock}}{d^G - d^B}.$$

Consumer Optimization: The Risk Dimension

After we use the prices of traded assets to infer the prices of contingent claims ...

... we can use the contingent claims prices to infer the price of any newly-introduced asset.

Black-Scholes Option Pricing

A call option is a contract that gives the buyer the right, but not the obligation, to purchase a share of stock at the strike price K at $t = 1$.

At $t = 1$, the call is said to be in the money if the actual share price is above the strike price and out of the money if the actual share price is below the strike price.

At $t = 1$, the option will have value only if it is in the money. But at $t = 0$, the option will have value even if there is only a probability of it being in the money at $t = 1$.

Black-Scholes Option Pricing

Fischer Black (US, 1938-1995) and Myron Scholes (Canada/US, b.1941, Nobel Prize 1997) were the first to derive a formula for the price of an option.

Robert Merton (US, b.1944, Nobel Prize 1997) arrived at the same formula in a simpler way, by showing how options prices could be inferred from assumptions about and observations on the underlying stock price.

Black-Scholes Option Pricing

The arguments used by Merton were not exactly those from Arrow-Debreu no-arbitrage theory that would use the price of the stock and bond to infer contingent claims prices, then use contingent claims prices to compute the price of the option.

But his analysis followed along similar lines, and today it is recognized that one could use the Arrow-Debreu approach to obtain the same results.

Black-Scholes Option Pricing

Their papers were both published in 1973.

Fischer Black and Myron Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* Vol.81 (May-June 1973): pp.637-654.

Robert Merton, "Theory of Rational Option Pricing," *The Bell Journal of Economics and Management Science* Vol.4 (Spring 1973): pp.141-183.

Black-Scholes Option Pricing

To see how the theory works, assume a simple two-period structure, with $t = 0$ and $t = 1$, and assume as well, that there are only two states, $i = G$ and $i = B$, at $t = 1$. Let

$$q^s = \text{ price of the stock at } t = 0$$

$$P^G = \text{ price of the stock in state } i = G \text{ at } t = 1$$

$$P^B = \text{ price of the stock in state } i = B \text{ at } t = 1$$

Black-Scholes Option Pricing

Likewise, let

$$q^b = \text{ price of the bond at } t = 0$$

$$1 = \text{ payoff from bond at } i = G \text{ at } t = 1$$

$$1 = \text{ payoff from bond at } i = B \text{ at } t = 1$$

Black-Scholes Option Pricing

Now consider a call option on the stock with strike price K. Let

$$q^o = \text{ price of the call at } t = 0$$

$$C^G = \text{ payoff generated by the call in state } i = G \text{ at } t = 1$$

$$C^B = \text{ payoff generated by the call in state } i = B \text{ at } t = 1$$

Assume, for now, that the call is in the money in both states at $t = 1$. Then:

$$C^G = P^G - K \text{ and } C^B = P^B - K$$

Black-Scholes Option Pricing

One of the key insights that underlies the Black-Scholes formula is that we don't need to make any specific assumptions about risk or risk aversion to price the option.

Instead, we can use a no-arbitrage argument that:

1. Replicates the option's payoffs using a portfolio of the stock and the risk-free bond.
2. Values the option based on the cost of assembling the portfolio.

Black-Scholes Option Pricing

State	Stock's Payoff	Bond's Payoff	Option's Payoff
G	P^G	1	$P^G - K$
B	P^B	1	$P^B - K$

We want to construct a portfolio consisting of s shares of the stock and b bonds that replicates the payoffs from the option in both states at $t = 1$:

$$sP^G + b = P^G - K$$

$$sP^B + b = P^B - K$$

Black-Scholes Option Pricing

$$sP^G + b = P^G - K$$
$$sP^B + b = P^B - K$$

This is a set of two linear equations in the two unknowns: s and b. The solution is

$$s = 1 \text{ and } b = -K$$

Since the stock costs q^s and the bond costs q^b, the cost of this portfolio at $t = 0$ is

$$q^s - q^b K$$

Black-Scholes Option Pricing

The option's payoffs are replicated by a portfolio with

$$s = 1 \text{ and } b = -K$$

and since the stock costs q^s and the bond costs q^b, the cost of this portfolio at $t = 0$ is

$$q^s - q^b K$$

But this means that the price of the option must also be

$$q^o = q^s - q^b K$$

Black-Scholes Option Pricing

Next, let's consider the case in which the call is in the money in the good state and out of the money in the bad state at $t = 1$.

Then

$$C^G = P^G - K \text{ and } C^B = 0$$

Black-Scholes Option Pricing

State	Stock's Payoff	Bond's Payoff	Option's Payoff
G	P^G	1	$P^G - K$
B	P^B	1	0

Again we want to construct a portfolio consisting of s shares of the stock and b bonds that replicates the payoffs from the option in both states at $t = 1$:

$$sP^G + b = P^G - K$$

$$sP^B + b = 0$$

Black-Scholes Option Pricing

$$sP^G + b = P^G - K$$

$$sP^B + b = 0$$

Again this is a set of two linear equations in the two unknowns: s and b. The solution is

$$s = \frac{P^G - K}{P^G - P^B} \text{ and } b = -\frac{P^B(P^G - K)}{P^G - P^B}$$

Since the stock costs q^s and the bond costs q^b, the cost of this portfolio at $t = 0$ is

$$\left(\frac{P^G - K}{P^G - P^B}\right) q^s + \left[-\frac{P^B(P^G - K)}{P^G - P^B}\right] q^b$$

Black-Scholes Option Pricing

But since the portfolio of the stock and bond again replicates the payoffs from the option, this implies that the option's price must be

$$\begin{aligned} q^o &= \left(\frac{P^G - K}{P^G - P^B}\right) q^s + \left[-\frac{P^B(P^G - K)}{P^G - P^B}\right] q^b \\ &= \frac{(q^s - q^b P^B)(P^G - K)}{P^G - P^B} \end{aligned}$$

Black-Scholes Option Pricing

Finally, there is the easy case in which the call is out of the money in both states at $t = 1$.

Then

$$C^G = 0 \text{ and } C^B = 0$$

The option's payoffs can be replicated by a portfolio consisting of zero shares of the stock and zero bonds, which costs zero at $t = 0$. Equivalently, an asset that pays off nothing should cost nothing.