ECON 337901
FINANCIAL ECONOMICS

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Constrained Optimization

$$L(x, \lambda) = F(x) + \lambda[c - G(x)],$$

Theorem (Kuhn-Tucker) If x^* maximizes $F(x)$ subject to $c \geq G(x)$, then there exists a value $\lambda^* \geq 0$ such that, together, x^* and λ^* satisfy the first-order condition

$$F'(x^*) - \lambda^* G'(x^*) = 0$$

and the complementary slackness condition

$$\lambda^*[c - G(x^*)] = 0.$$

Constrained Optimization

In the case where $c > G(x^*)$, the constraint is non-binding. The complementary slackness condition

$$\lambda^*[c - G(x^*)] = 0$$

requires that $\lambda^* = 0$.

And the first-order condition

$$F'(x^*) - \lambda^* G'(x^*) = 0$$

requires that $F'(x^*) = 0$.

Constrained Optimization

In the case where $c = G(x^*)$, the constraint is binding. The complementary slackness condition

$$\lambda^*[c - G(x^*)] = 0$$

puts no further restriction on $\lambda^* \geq 0$.

Now the first-order condition

$$F'(x^*) - \lambda^* G'(x^*) = 0$$

requires that $F'(x^*) = \lambda^* G'(x^*)$.

Constrained Optimization: Example 1

For the problem

$$\max_x \left(-\frac{1}{2}\right)(x-5)^2 \text{ subject to } 7 \geq x,$$

$F(x) = (-1/2)(x-5)^2$, $c = 7$, and $G(x) = x$. The Lagrangian is

$$L(x, \lambda) = \left(-\frac{1}{2}\right)(x-5)^2 + \lambda(7-x).$$

Constrained Optimization: Example 1

With

$$L(x, \lambda) = \left(-\frac{1}{2}\right)(x-5)^2 + \lambda(7-x),$$

the first-order condition

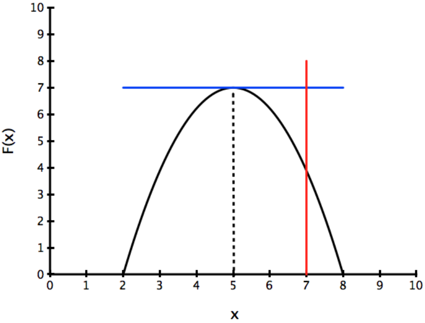
$$-(x^* - 5) - \lambda^* = 0$$

and the complementary slackness condition

$$\lambda^*(7 - x^*) = 0$$

are satisfied with $x^* = 5$, $F'(x^*) = 0$, $\lambda^* = 0$, and $7 > x^*$.

Constrained Optimization: Example 1



Here, the solution has $F'(x^*) = 0$ since the constraint is nonbinding.

Constrained Optimization: Example 2

For the problem

$$\max_x \left(-\frac{1}{2}\right)(x-5)^2 \text{ subject to } 4 \geq x,$$

$F(x) = (-1/2)(x-5)^2$, $c = 4$, and $G(x) = x$. The Lagrangian is

$$L(x,\lambda) = \left(-\frac{1}{2}\right)(x-5)^2 + \lambda(4-x).$$

Constrained Optimization: Example 2

With

$$L(x, \lambda) = \left(-\frac{1}{2}\right)(x-5)^2 + \lambda(4-x),$$

the first-order condition

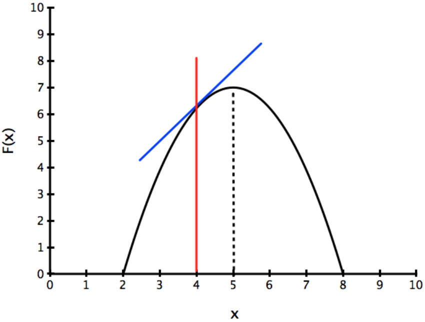
$$-(x^* - 5) - \lambda^* = 0$$

and the complementary slackness condition

$$\lambda^*(4 - x^*) = 0$$

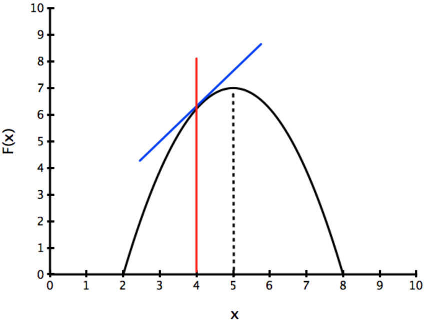
are satisfied with $x^* = 4$ and $F'(x^*) = \lambda^* = 1 > 0$.

Constrained Optimization: Example 2



Here, the solution has $F'(x^*) = \lambda^* G'(x^*) > 0$ since the constraint is binding. $F'(x^*) > 0$ indicates that we'd like to increase the value of x, but the constraint won't let us.

Constrained Optimization: Example 2



With a binding constraint, $F'(x^*) \neq 0$ but $F'(x^*) - \lambda^* G'(x^*) = 0$. The value x^* that solves the problem is a critical point, not of the objective function $F(x)$, but instead of the entire Lagrangian $F(x) + \lambda[c - G(x)]$.