Constrained Optimization

\[ L(x, \lambda) = F(x) + \lambda [c - G(x)], \]

**Theorem (Kuhn-Tucker)** If \( x^* \) maximizes \( F(x) \) subject to \( c \geq G(x) \), then there exists a value \( \lambda^* \geq 0 \) such that, together, \( x^* \) and \( \lambda^* \) satisfy the **first-order condition**

\[ F'(x^*) - \lambda^* G'(x^*) = 0 \]

and the **complementary slackness condition**

\[ \lambda^* [c - G(x^*)] = 0. \]
Constrained Optimization

In the case where \( c > G(x^*) \), the constraint is non-binding. The complementary slackness condition

\[
\lambda^*[c - G(x^*)] = 0
\]

requires that \( \lambda^* = 0 \).

And the first-order condition

\[
F'(x^*) - \lambda^*G'(x^*) = 0
\]

requires that \( F'(x^*) = 0 \).
Constrained Optimization

In the case where \( c = G(x^*) \), the constraint is binding. The complementary slackness condition

\[
\lambda^* [c - G(x^*)] = 0
\]

puts no further restriction on \( \lambda^* \geq 0 \).

Now the first-order condition

\[
F'(x^*) - \lambda^* G'(x^*) = 0
\]

requires that \( F'(x^*) = \lambda^* G'(x^*) \).
Constrained Optimization: Example 1

For the problem

\[
\max_x \left( -\frac{1}{2} \right) (x - 5)^2 \text{ subject to } 7 \geq x,
\]

\[ F(x) = (-1/2)(x - 5)^2, \quad c = 7, \quad \text{and } G(x) = x. \]

The Lagrangian is

\[
L(x, \lambda) = \left( -\frac{1}{2} \right) (x - 5)^2 + \lambda(7 - x).
\]
Constrained Optimization: Example 1

With

\[ L(x, \lambda) = \left( -\frac{1}{2} \right) (x - 5)^2 + \lambda (7 - x), \]

the first-order condition

\[ -(x^* - 5) - \lambda^* = 0 \]

and the complementary slackness condition

\[ \lambda^* (7 - x^*) = 0 \]

are satisfied with \( x^* = 5, \ F'(x^*) = 0, \ \lambda^* = 0, \) and \( 7 > x^*. \)
Constrained Optimization: Example 1

Here, the solution has $F'(x^*) = 0$ since the constraint is nonbinding.
Constrained Optimization: Example 2

For the problem

\[
\max_x \left( -\frac{1}{2} \right) (x - 5)^2 \text{ subject to } 4 \geq x,
\]

\( F(x) = (-1/2)(x - 5)^2, \ c = 4, \) and \( G(x) = x. \) The Lagrangian is

\[
L(x, \lambda) = \left( -\frac{1}{2} \right) (x - 5)^2 + \lambda(4 - x).
\]
Constrained Optimization: Example 2

With

\[ L(x, \lambda) = \left(-\frac{1}{2}\right)(x - 5)^2 + \lambda(4 - x), \]

the first-order condition

\[ -(x^* - 5) - \lambda^* = 0 \]

and the complementary slackness condition

\[ \lambda^*(4 - x^*) = 0 \]

are satisfied with \( x^* = 4 \) and \( F'(x^*) = \lambda^* = 1 > 0 \).
Constrained Optimization: Example 2

Here, the solution has $F'(x^*) = \lambda^* G'(x^*) > 0$ since the constraint is binding. $F'(x^*) > 0$ indicates that we’d like to increase the value of $x$, but the constraint won’t let us.
With a binding constraint, $F'(x^*) \neq 0$ but $F'(x^*) - \lambda^* G'(x^*) = 0$. The value $x^*$ that solves the problem is a critical point, not of the objective function $F(x)$, but instead of the entire Lagrangian $F(x) + \lambda[c - G(x)]$. 