

ECON 337901

FINANCIAL ECONOMICS

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Constrained Optimization

To find the value of x that solves

$$\max_x F(x) \text{ subject to } c \geq G(x)$$

you can:

1. Try out every possible value of x .
2. Use calculus.

Since search could take forever, let's use calculus instead.

Constrained Optimization

A method for solving constrained optimization problems like

$$\max_x F(x) \text{ subject to } c \geq G(x)$$

was developed by Joseph-Louis Lagrange (France/Italy, 1736-1813) and extended by Harold Kuhn (US, 1925-2014) and Albert Tucker (US, 1905-1995).

Constrained Optimization

Associated with the problem:

$$\max_x F(x) \text{ subject to } c \geq G(x)$$

Define the **Lagrangian**

$$L(x, \lambda) = F(x) + \lambda[c - G(x)],$$

where λ is the **Lagrange multiplier**.

Constrained Optimization

Then, look for a critical point of the full Lagrangian

$$L(x, \lambda) = F(x) + \lambda[c - G(x)],$$

instead of just the objective function F by itself.

That is, use the FOC

$$F'(x^*) - \lambda^* G'(x^*) = 0.$$

Constrained Optimization

$$L(x, \lambda) = F(x) + \lambda[c - G(x)],$$

Theorem (Kuhn-Tucker) If x^* maximizes $F(x)$ subject to $c \geq G(x)$, then there exists a value $\lambda^* \geq 0$ such that, together, x^* and λ^* satisfy the **first-order condition**

$$F'(x^*) - \lambda^* G'(x^*) = 0$$

and the **complementary slackness condition**

$$\lambda^*[c - G(x^*)] = 0.$$

Constrained Optimization

In the case where $c > G(x^*)$, the constraint is **non-binding**.
The complementary slackness condition

$$\lambda^*[c - G(x^*)] = 0$$

requires that $\lambda^* = 0$.

And the first-order condition

$$F'(x^*) - \lambda^* G'(x^*) = 0$$

requires that $F'(x^*) = 0$.

Constrained Optimization

In the case where $c = G(x^*)$, the constraint is **binding**. The complementary slackness condition

$$\lambda^*[c - G(x^*)] = 0$$

puts no further restriction on $\lambda^* \geq 0$.

Now the first-order condition

$$F'(x^*) - \lambda^* G'(x^*) = 0$$

requires that $F'(x^*) = \lambda^* G'(x^*)$.