ECON 337901
FINANCIAL ECONOMICS

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Mathematical Preliminaries

Unconstrained Optimization

$$\max_x F(x)$$

Constrained Optimization

$$\max_x F(x) \text{ subject to } c \geq G(x)$$

Unconstrained Optimization

To find the value of x that solves

$$\max_x F(x)$$

you can:

1. Try out every possible value of x.
2. Use calculus.

Since search could take forever, let's use calculus instead.

Unconstrained Optimization

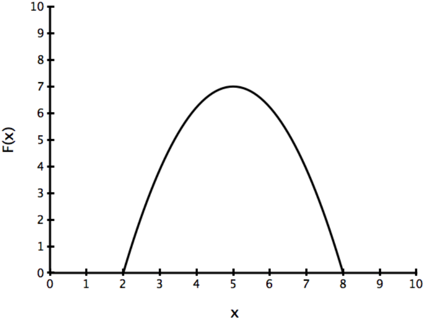
Theorem If x^* solves

$$\max_x F(x),$$

then x^* is a critical point of F, that is,

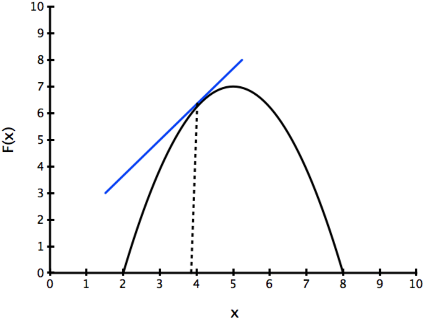
$$F'(x^*) = 0.$$

Unconstrained Optimization



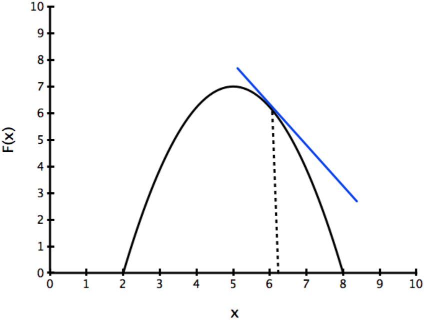
$F(x)$ maximized at $x^* = 5$

Unconstrained Optimization



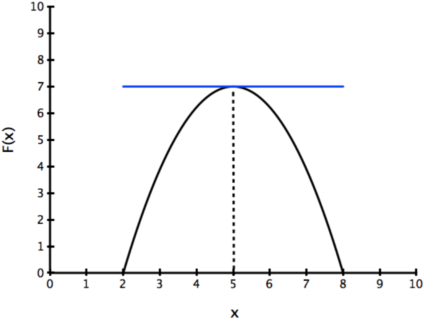
$F'(x) > 0$ when $x < 5$. $F(x)$ can be increased by increasing x.

Unconstrained Optimization



$F'(x) < 0$ when $x > 5$. $F(x)$ can be increased by decreasing x.

Unconstrained Optimization



$F'(x) = 0$ when $x = 5$. $F(x)$ is maximized.

Unconstrained Optimization

Theorem If x^* solves

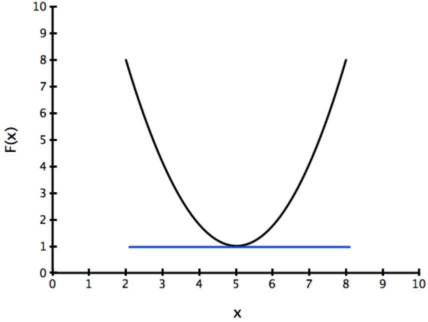
$$\max_x F(x),$$

then x^* is a critical point of F, that is,

$$F'(x^*) = 0.$$

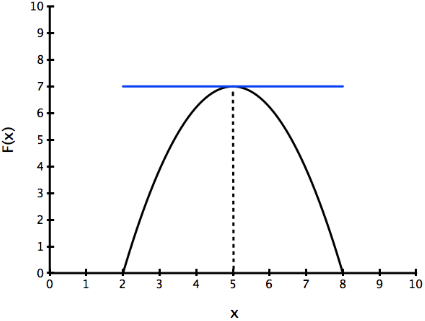
Note that the same first-order necessary condition $F'(x^*) = 0$ also characterizes a value of x^* that minimizes $F(x)$.

Unconstrained Optimization



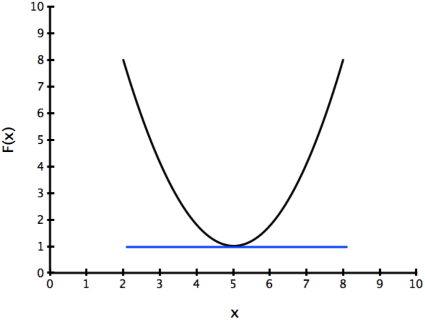
$F'(x) = 0$ when $x = 5$. $F(x)$ is minimized.

Unconstrained Optimization



$F'(x) = 0$ and $F''(x) < 0$ when $x = 5$. $F(x)$ is maximized.

Unconstrained Optimization



$F'(x) = 0$ and $F''(x) > 0$ when $x = 5$. $F(x)$ is minimized.

Unconstrained Optimization

Theorem If x^* solves

$$\max_x F(x),$$

then x^* is a critical point of F, that is,

$$F'(x^*) = 0.$$

Unconstrained Optimization

Theorem If

$$F'(x^*) = 0 \text{ and } F''(x^*) < 0,$$

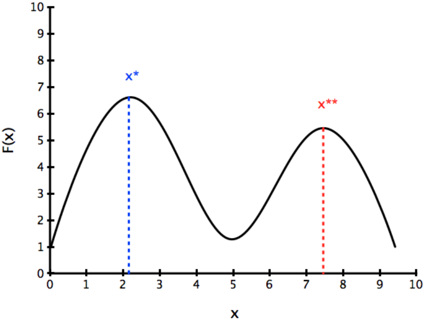
then x^* solves

$$\max_x F(x)$$

(at least locally).

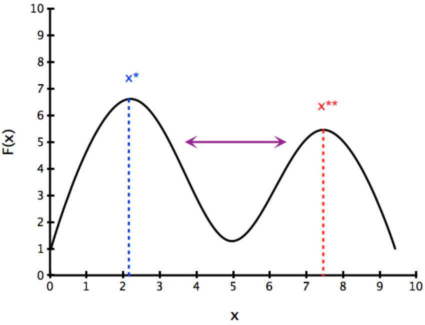
The first-order condition $F'(x^*) = 0$ and the second-order condition $F''(x^*) < 0$ are sufficient conditions for the value of x that (locally) maximizes $F(x)$.

Unconstrained Optimization



$F'(x^{**}) = 0$ and $F''(x^{**}) < 0$ at the local maximizer x^{**} and
$F'(x^{*}) = 0$ and $F''(x^{*}) < 0$ at the global maximizer x^{*}.

Unconstrained Optimization



$F'(x^{**}) = 0$ and $F''(x^{**}) < 0$ at the local maximizer x^{**} and $F'(x^{*}) = 0$ and $F''(x^{*}) < 0$ at the global maximizer x^{*}, but $F''(x) > 0$ in between x^{*} and x^{**}.

Unconstrained Optimization

Theorem If

$$F'(x^*) = 0$$

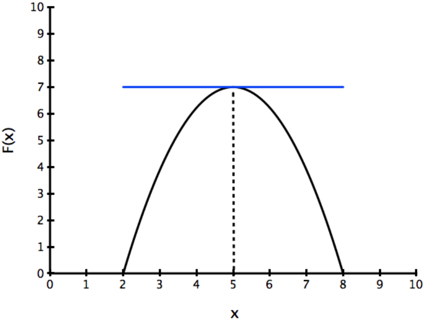
and

$$F''(x) < 0 \text{ for all } x \in \mathbb{R},$$

then x^* solves

$$\max_x F(x).$$

Unconstrained Optimization



$F''(x) < 0$ for all $x \in \mathbb{R}$ and $F'(5) = 0$. $F(x)$ is maximized when $x = 5$.

Unconstrained Optimization

If $F''(x) < 0$ for all $x \in \mathbb{R}$, then the function F is concave.

When F is concave, the first-order condition $F'(x^*) = 0$ is both necessary and sufficient for the value of x that maximizes $F(x)$.

And, as we are about to see, concave functions arise frequently and naturally in economics and finance.

Unconstrained Optimization: Example 1

Consider the problem

$$\max_{x} \left(-\frac{1}{2}\right)(x-\tau)^2,$$

where τ is a number ($\tau \in \mathbb{R}$) that we might call the "target."

The first-order condition

$$-(x^* - \tau) = 0$$

leads us immediately to the solution: $x^* = \tau$.

Unconstrained Optimization: Example 2

Consider maximizing a function of three variables:

$$\max_{x_1, x_2, x_3} F(x_1, x_2, x_3)$$

Even if each variable can take on only 1,000 values, there are one billion possible combinations of (x_1, x_2, x_3) to search over!

This is an example of what Richard Bellman (US, 1920-1984) called the "curse of dimensionality."

Unconstrained Optimization: Example 2

Consider the problem:

$$\max_{x_1,x_2,x_3} \left(-\frac{1}{2}\right)(x_1-\tau)^2 + \left(-\frac{1}{2}\right)(x_2-x_1)^2 + \left(-\frac{1}{2}\right)(x_3-x_2)^2.$$

Now the three first-order conditions

$$-(x_1^*-\tau)+(x_2^*-x_1^*)=0$$

$$-(x_2^*-x_1^*)+(x_3^*-x_2^*)=0$$

$$-(x_3^*-x_2^*)=0$$

lead us to the solution: $x_1^*=x_2^*=x_3^*=\tau$.