ECON 337901
FINANCIAL ECONOMICS

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March 21, 2019

3 Making Choices in Risky Situations

A Criteria for Choice Over Risky Prospects ✔
B Preferences and Utility Functions ✔
C Expected Utility Functions
D The Expected Utility Theorem
E The Allais Paradox

Next: Measuring Risk and Risk Aversion

Expected Utility Functions

Under certainty, the "goods" are described by consumption baskets with known characteristics.

Under uncertainty, the "goods" are random (state-contingent) payoffs.

The problem of describing preferences over these state-contingent payoffs, and then summarizing these preferences with a utility function, is similar in overall spirit but somewhat different in its details to the problem of describing preferences and utility functions under certainty.

Expected Utility Functions

Consider shares of stock in two companies:

		Price Next Year in	
	Price Today	Good State	Bad State
AT&T	-100	150	100
Verizon	-100	150	100

where the good state occurs with probability π and the bad state occurs with probability $1-\pi$.

Expected Utility Functions

		Price Next Year in	
	Price Today	Good State	Bad State
AT&T	-100	150	100
Verizon	-100	150	100
	probability	π	$1-\pi$

We will assume that if the two assets provide exactly the same state-contingent payoffs, then investors will be indifferent between them.

Expected Utility Functions

		Price Next Year in	
	Price Today	Good State	Bad State
AT&T	-100	150	100
Verizon	-100	150	100
	probability	π	$1-\pi$

1. Investors care only about payoffs and probabilities.

Expected Utility Functions

Consider another comparison:

	Price Today	Price Next Year in Good State	Bad State
AT&T	-100	150	100
Apple	-100	160	110
	probability	π	$1-\pi$

We will also assume that investors will prefer any asset that exhibits state-by-state dominance over another.

Expected Utility Functions

		Price Next Year in	
	Price Today	Good State	Bad State
AT&T	-100	150	100
Apple	-100	160	110
	probability	π	$1-\pi$

2. If $u(p)$ measures utility from the payoff p in any particular state, then u is increasing.

Expected Utility Functions

Consider a third comparison:

		Price Next Year in	
	Price Today	Good State	Bad State
AT&T	-100	150	100
IBM	-100	160	90
	probability	π	$1-\pi$

Here, there is no state-by-state dominance, but it seems reasonable to assume that a higher probability π will make investors tend to prefer IBM, while a higher probability $1-\pi$ will make investors tend to prefer AT&T.

Expected Utility Functions

		Price Next Year in	
	Price Today	Good State	Bad State
AT&T	-100	150	100
IBM	-100	160	90
	probability	π	$1-\pi$

3. Investors should care more about states of the world that occur with greater probability.

Expected Utility Functions

A criterion that has all three of these properties was suggested by Blaise Pascal (France, 1623-1662): base decisions on the expected payoff,

$$E(p) = \pi p_G + (1 - \pi) p_B,$$

where p_G and p_B, with $p_G > p_B$, are the payoffs in the good and bad states.

Expected Utility Functions

Expected payoff

$$E(p) = \pi p_G + (1 - \pi) p_B$$

1. Depends only on payoffs and probabilities.
2. Increases whenever p_G or p_B rises.
3. Attaches higher weight to states with higher probabilities.

Expected Utility Functions

Nicolaus Bernoulli (Switzerland, 1687-1759) pointed to a problem with basing investment decisions exclusively on expected payoffs: it ignores risk. To see this, specialize the previous example by setting $\pi = 1 - \pi = 1/2$ but add, as well, a third asset:

		Price Next Year in	
	Price Today	Good State	Bad State
AT&T	-100	150	100
IBM	-100	160	90
US Gov't Bond	-100	125	125
	probability	$\pi = 1/2$	$1 - \pi = 1/2$

Expected Utility Functions

		Price Next Year in	
	Price Today	Good State	Bad State
AT&T	-100	150	100
IBM	-100	160	90
US Gov't Bond	-100	125	125
	probability	$\pi = 1/2$	$1 - \pi = 1/2$

AT&T: $E(p) = (1/2)150 + (1/2)100 = 125$

IBM $E(p) = (1/2)160 + (1/2)90 = 125$

Gov't Bond: $E(p) = (1/2)125 + (1/2)125 = 125$

Expected Utility Functions

$$\text{AT\&T: } E(p) = (1/2)150 + (1/2)100 = 125$$

$$\text{IBM } E(p) = (1/2)160 + (1/2)90 = 125$$

$$\text{Gov't Bond: } E(p) = (1/2)125 + (1/2)125 = 125$$

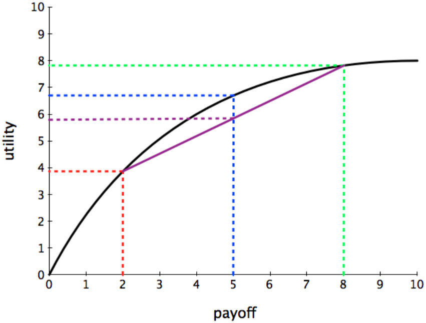
All three assets have the same expected payoff, but the bond is less risky than both stocks and AT&T stock is less risky than IBM stock.

Expected Utility Functions

Gabriel Cramer (Switzerland, 1704-1752) and Daniel Bernoulli (Switzerland, 1700-1782) suggested that more reliable comparisons could be made by assuming that the utility function u over payoffs in any given state is concave as well as increasing.

This implies that investors prefer more to less, but have diminishing marginal utility as payoffs increase.

Expected Utility Functions



When u is concave, a payoff of 5 for sure is preferred to a payoff of 8 with probability $1/2$ and 2 with probability $1/2$.

Expected Utility Functions

About two centuries later, John von Neumann (Hungary, 1903-1957) and Oskar Morgenstern (Germany, 1902-1977) worked out the conditions under which investors' preferences over risky payoffs could be described by an expected utility function such as

$$U(p) = E[u(p)] = \pi u(p_G) + (1 - \pi) u(p_B),$$

where the Bernoulli utility function over payoffs u is increasing and concave and the von Neumann-Morgenstern expected utility function U is linear in the probabilities.

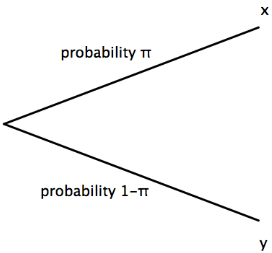
Expected Utility Functions

von Neumann and Morgenstern's axiomatic derivation of expected utility appeared in the second edition of their book, *Theory of Games and Economic Behavior*, published in 1947.

$$U(p) = E[u(p)] = \pi u(p_G) + (1 - \pi)u(p_B)$$

Linearity in the probabilities is the "defining characteristic" of the expected utility function $U(p)$.

The Expected Utility Theorem



The simple lottery (x, y, π) offers payoff x with probability π and payoff y with probability $1 - \pi$.

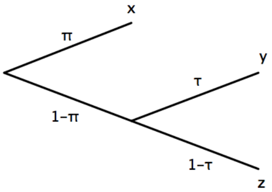
The Expected Utility Theorem

The simple lottery (x, y, π) offers payoff x with probability π and payoff y with probability $1 - \pi$.

In this definition, x and y can be monetary payoffs, as in the stock and bond examples from before.

Alternatively, they can be additional lotteries!

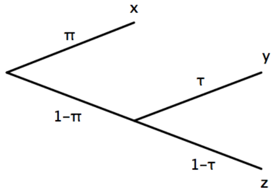
The Expected Utility Theorem



The compound lottery $(x, (y, z, \tau), \pi)$ offers payoff x with probability π and lottery (y, z, τ) with probability $1 - \pi$.

The Expected Utility Theorem

Notice that a simple lottery with more than two outcomes can always be reinterpreted as a compound lottery where each individual lottery has only two outcomes.



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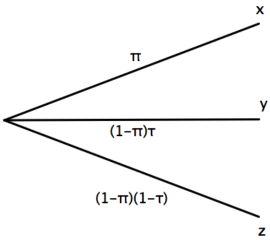

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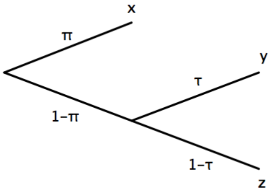

The Expected Utility Theorem

Notice that a simple lottery with more than two outcomes can always be reinterpreted as a compound lottery where each individual lottery has only two outcomes.

So restricting ourselves to lotteries with only two outcomes does not entail any loss of generality in terms of the number of future states that are possible.

But to begin describing preferences over lotteries, we need to make additional assumptions.

The Expected Utility Theorem

B1a A lottery that pays off x with probability one is the same as getting x for sure: $(x, y, 1) = x$.

B1b Investors care about payoffs and probabilities, but not the specific ordering of the states: $(x, y, \pi) = (y, x, 1 - \pi)$

B1c In evaluating compound lotteries, investors care only about the probabilities of each final payoff:
$(x, z, \pi) = (x, y, \pi + (1 - \pi)\tau)$ if $z = (x, y, \tau)$.

The Expected Utility Theorem

B2 There exists a preference relation $\succeq$ defined on lotteries that is complete and transitive.

Again, this amounts to requiring that investors are fully informed and rational.

The Expected Utility Theorem

B3 The preference relation $\succeq$ defined on lotteries is continuous.

Hence, very small changes in lotteries cannot lead to very large changes in preferences over those lotteries.

The Expected Utility Theorem

By the previous theorem, we already know that (B2) and (B3) are sufficient to guarantee the existence of a utility function over lotteries and, by (B1a), payoffs received with certainty as well.

What remains is to identify the extra assumptions that guarantee that this utility function is linear in the probabilies, that is, of the von Neumann-Morgenstern (vN-M) form.

The Expected Utility Theorem

B4 Independence axiom: For any two lotteries (x, y, π) and (x, z, π), $y \succeq z$ if and only if $(x, y, \pi) \succeq (x, z, \pi)$.

This assumption is controversial and unlike any made in traditional microeconomic theory: you would not necessary want to assume that a consumer's preferences over sub-bundles of any two goods are independent of how much of a third good gets included in the overall bundle. But it is needed for the utility function to take the vN-M form.

The Expected Utility Theorem

Theorem (Expected Utility Theorem) If (B1)-(B4) hold, then there exists a utility function U defined over lotteries of the von Neumann-Morgenstern form

$$U((x, y, \pi)) = \pi u(x) + (1 - \pi)u(y).$$

that represents the underlying preference relation in the sense that for any two lotteries z and z',

$$z \succeq z' \text{ if and only if } U(z) \geq U(z').$$

The Expected Utility Theorem

Note that the key property of the vN-M utility function

$$U(z) = U(x, y, \pi) = \pi u(x) + (1 - \pi)u(y),$$

its linearity in the probabilities π and $1 - \pi$, is not preserved by all transformations of the form

$$V(z) = F(U(z)),$$

where F is an increasing function.

In this sense, vN-M utility functions are cardinal, not ordinal.

The Expected Utility Theorem

On the other hand, given a vN-M utility function

$$U(z) = U(x, y, \pi) = \pi u(x) + (1 - \pi)u(y),$$

consider an affine transformation

$$V(z) = \alpha U(z) + \beta$$

and define

$$v(x) = \alpha u(x) + \beta \text{ and } v(y) = \alpha u(y) + \beta$$

The Expected Utility Theorem

$$U(z) = U(x, y, \pi) = \pi u(x) + (1 - \pi)u(y),$$

$$V(z) = \alpha U(z) + \beta.$$

$$v(x) = \alpha u(x) + \beta \text{ and } v(y) = \alpha u(y) + \beta$$

$$\begin{aligned} V(x, y, \pi) &= \alpha U(x, y, \pi) + \beta \\ &= \alpha[\pi u(x) + (1 - \pi)u(y)] + \beta \\ &= \pi[\alpha u(x) + \beta] + (1 - \pi)[\alpha u(y) + \beta] \\ &= \pi v(x) + (1 - \pi)v(y). \end{aligned}$$

In this sense, the vN-M utility function that represents any given preference relation is not unique.

Problem Set 9, Question 2

Consider an investor with initial wealth $W_0 = 10$, whose preferences over simple lotteries (x, y, π) can be described by the von Neumann-Morgenstern Expected Utility function

$$U(x, y, \pi) = \pi u(W_0 + x) + (1 - \pi) u(W_0 + y)$$

where the Bernoulli utility function takes the special form

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

with $\gamma > 0$.

Problem Set 9, Question 2

In the early 1700s, Daniel Bernoulli suggested that concavity of the utility function over payoffs could capture the element of risk aversion that Pascal's criterion – base decisions on expected payoff alone – missed.

With

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

$$u'(c) = c^{-\gamma} > 0 \text{ and } u''(c) = -\gamma c^{-\gamma-1} < 0$$

so that the Bernoulli utility function gets "more concave" as γ rises. Does this also make the investor more risk averse?

Problem Set 9, Question 2

To explore this possibility, consider three lotteries:

L1) $(x, y, \pi) = (5, 0, 1/2) =$ coin flip for 5 or 0

L2) $(x, y, \pi) = (2.5, 0, 1) =$ 2.50 for sure

L3) $(x, y, \pi) = (2, 0, 1) =$ 2 for sure

Problem Set 9, Question 2

L1) $(x, y, \pi) = (5, 0, 1/2) =$ coin flip for 5 or 0

L2) $(x, y, \pi) = (2.5, 0, 1) =$ 2.50 for sure

L3) $(x, y, \pi) = (2, 0, 1) =$ 2 for sure

Every risk averse investor will choose L2 over L1 and L3.

But the choice between L1 and L3 will depend on risk aversion. Very risk averse investors will prefer L3 to L1 but less risk aversion investors may prefer L1 to L2.

Problem Set 9, Question 2

For each lottery (x, y, π), calculate expected utility

$$U(x, y, \pi) = \pi \left[\frac{(10 + x)^{1-\gamma}}{1 - \gamma} \right] + (1 - \pi) \left[\frac{(10 + y)^{1-\gamma}}{1 - \gamma} \right]$$

for an investor with $\gamma = 1/2$. Rank the lotteries from most to least preferred, based on expected utility.

Then repeat the exercise for $\gamma = 2$ and $\gamma = 3$.

Problem Set 9, Question 2

L1) $(x, y, \pi) = (5, 0, 1/2) =$ coin flip for 5 or 0

L2) $(x, y, \pi) = (2.5, 0, 1) = 2.50$ for sure

L3) $(x, y, \pi) = (2, 0, 1) = 2$ for sure

All three investors get their highest expected utility from L2.

But the choice between L1 and L3 depends on risk aversion. As γ rises, preferences shift from L1 to L3 as the second choice.

Problem Set 9, Question 2

Is it always the case that making $u''(c)$ "more negative" so that the Bernoulli utility function becomes "more concave" makes an investor more risk averse?

Unfortunately, no.

Problem Set 9, Question 2

Suppose an investor's preferences are described by the expected utility function

$$U(x, y, \pi) = \pi u(x) + (1 - \pi)u(y)$$

in the sense that

$$(x, y, \pi) \succeq (x', y', \pi') \text{ if and only if } U(x, y, \pi) \geq U(x', y', \pi')$$

Then those same preferences are described equally well by the expected utility function

$$V(x, y, \pi) = \alpha U(x, y, \pi)$$

for any value of $\alpha > 0$.

Problem Set 9, Question 2

Since

$$\begin{aligned} V(x, y, \pi) &= \alpha U(x, y, \pi) \\ &= \alpha[\pi u(x) + (1-\pi)u(y)] \\ &= \alpha\pi u(x) + \alpha(1-\pi)u(y) \\ &= \pi\alpha u(x) + (1-\pi)\alpha u(y) \\ &= \pi v(x) + (1-\pi)v(y) \end{aligned}$$

where

$$v(c) = \alpha u(c) \implies v'(c) = \alpha u'(c) \implies v''(c) = \alpha u''(c)$$

we can make $v''(c)$ "more" or "less" concave by choosing larger or smaller values of α, without changing the underlying preference ordering.