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FINANCIAL ECONOMICS

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Consumer Optimization: The Risk Dimension

Do we really observe consumers trading in contingent claims?

Yes, if we think of financial assets as "bundles" of contingent claims.

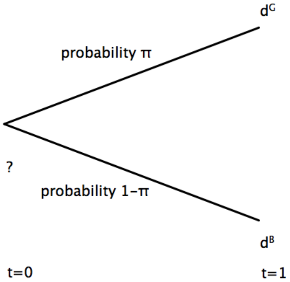
This insight is also Arrow and Debreu's.

Consumer Optimization: The Risk Dimension

A "stock" is a risky asset that pays dividend d^G next year in the good state and d^B next year in the bad state.

These payoffs can be replicated by buying d^G contingent claims for the good state and d^B contingent claims for the bad state.

Consumer Optimization: The Risk Dimension



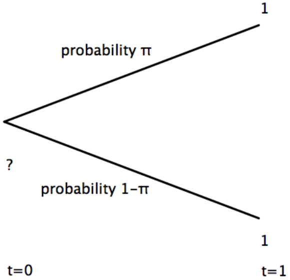
Payoffs for the stock.

Consumer Optimization: The Risk Dimension

A “bond” is a safe asset that pays off one next year in the good state and one next year in the bad state.

These payoffs can be replicated by buying one contingent claim for the good state and one contingent claim for the bad state.

Consumer Optimization: The Risk Dimension



Payoffs for the bond.

Consumer Optimization: The Risk Dimension

If we start with knowledge of the contingent claims prices q^G and q^B, then we can infer that the stock must sell today for

$$q^{stock} = q^G d^G + q^B d^B.$$

Since if the stock cost more than the equivalent bundle of contingent claims, traders could make profits for sure by short selling the stock and buying the contingent claims; and if the stock cost less than the equivalent bundle of contingent claims, traders could make profits for sure by buying the stock and selling the contingent claims.

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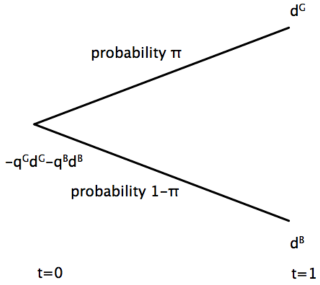


"Pricing" the stock.

Consumer Optimization: The Risk Dimension

Likewise, if we start with knowledge of the contingent claims prices q^G and q^B, then we can infer that the bond must sell today for

$$q^{bond} = q^G + q^B.$$

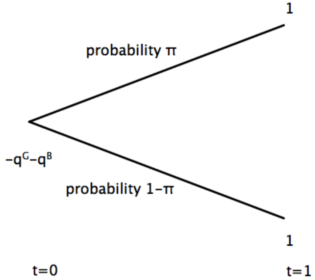
Since the bond pays off one for sure next year, the interest rate, defined as the return on the risk-free bond, is

$$1 + r = \frac{1}{q^{bond}} = \frac{1}{q^G + q^B}.$$

The bond price relates to the interest rate via

$$q^{bond} = \frac{1}{1 + r}.$$

Consumer Optimization: The Risk Dimension



Pricing the bond.

Consumer Optimization: The Risk Dimension

We've already seen how contingent claims can be used to replicate the stock and the bond.

Now let's see how the stock and the bond can be used to replicate the contingent claims.

Consumer Optimization: The Risk Dimension

Consider buying s shares of stock and b bonds, in order to replicate the contingent claim for the good state.

In the good state, the payoffs should be

$$sd^G + b = 1$$

and in the bad state, the payoffs should be

$$sd^B + b = 0$$

since the contingent claim pays off one in the good state and zero in the bad state.

Consumer Optimization: The Risk Dimension

To replicate the contingent claim for the good state:

$$sd^G + b = 1$$

$$sd^B + b = 0 \Rightarrow b = -sd^B$$

Substitute the second equation into the first to solve for

$$s = \frac{1}{d^G - d^B} \text{ and } b = \frac{-d^B}{d^G - d^B}$$

Since s and b are of opposite sign, this requires going "long" one asset and "short" the other.

Consumer Optimization: The Risk Dimension

To replicate the contingent claim for the good state:

$$s = \frac{1}{d^G - d^B} \text{ and } b = \frac{-d^B}{d^G - d^B}$$

If we know the prices q^{stock} and q^{bond} of the stock and bond, we can infer that in the absence of arbitrage, the claim for the good state would have price

$$q^G = q^{stock}s + q^{bond}b = \frac{q^{stock} - d^B q^{bond}}{d^G - d^B}.$$

Consumer Optimization: The Risk Dimension

Consider buying s shares of stock and b bonds, in order to replicate the contingent claim for the bad state.

In the good state, the payoffs should be

$$sd^G + b = 0$$

and in the bad state, the payoffs should be

$$sd^B + b = 1$$

since the contingent claim pays off one in the bad state and zero in the good state.

Consumer Optimization: The Risk Dimension

To replicate the contingent claim for the bad state:

$$sd^G + b = 0 \Rightarrow b = -sd^G$$

$$sd^B + b = 1$$

Substitute the first equation into the second to solve for

$$s = \frac{-1}{d^G - d^B} \text{ and } b = \frac{d^G}{d^G - d^B}$$

Once again, this requires going long one asset and short the other.

Consumer Optimization: The Risk Dimension

To replicate the contingent claim for the bad state:

$$s = \frac{-1}{d^G - d^B} \text{ and } b = \frac{d^G}{d^G - d^B}$$

Once again, if we know the prices q^{stock} and q^{bond} of the stock and bond, we can infer that in the absence of arbitrage, the claim for the bad state would have price

$$q^B = q^{stock}s + q^{bond}b = \frac{d^G q^{bond} - q^{stock}}{d^G - d^B}.$$