ECON 337901
FINANCIAL ECONOMICS

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Consumer Optimization: The Time Dimension

The problem is to choose c_0 and c_1 to maximize utility

$$u(c_0) + \beta u(c_1)$$

subject to the budget constraint

$$Y_0 + \frac{Y_1}{1+r} \geq c_0 + \frac{c_1}{1+r}.$$

The Lagrangian is

$$L(c_0, c_1, \lambda) = u(c_0) + \beta u(c_1) + \lambda \left(Y_0 + \frac{Y_1}{1+r} - c_0 - \frac{c_1}{1+r} \right).$$

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$$L(c_0, c_1, \lambda) = u(c_0) + \beta u(c_1) + \lambda \left(Y_0 + \frac{Y_1}{1+r} - c_0 - \frac{c_1}{1+r} \right)$$

The first-order conditions

$$u'(c_0^*) - \lambda^* = 0$$

$$\beta u'(c_1^*) - \lambda^* \left(\frac{1}{1+r} \right) = 0.$$

lead directly to the graphical result

$$\frac{u'(c_0^*)}{\beta u'(c_1^*)} = 1 + r.$$

Consumer Optimization: The Time Dimension

Now set $Y_0 = Y$, $Y_1 = 0$, and assume that $u(c) = \ln(c)$.

The problem specializes to

$$\max_{c_0, c_1} \ln(c_0) + \beta \ln(c_1) \text{ subject to } Y \geq c_0 + \frac{c_1}{1+r}.$$

With Lagrangian

$$L(c_0, c_1, \lambda) = \ln(c_0) + \beta \ln(c_1) + \lambda \left(Y - c_0 - \frac{c_1}{1+r} \right).$$

Consumer Optimization: The Time Dimension

$$L(c_0, c_1, \lambda) = \ln(c_0) + \beta \ln(c_1) + \lambda \left(Y - c_0 - \frac{c_1}{1+r} \right)$$

1. Write down the FOCs for c_0^* and c_1^*.
2. Divide one FOC by the other to eliminate λ^*.
3. Re-organize the result to see how consumption growth c_1^*/c_0^* depends on β and $1+r$.
4. How does c_1^*/c_0^* change when the consumer becomes more patient (β increases) and when the interest rate r rises?

Consumer Optimization: The Time Dimension

At first glance, Fisher's model seems unrealistic, especially in its assumption that the consumer can borrow at the same interest rate r that he or she receives on his or her savings.

A reinterpretation of saving and borrowing in this framework, however, can make it more applicable, at least for some consumers.

Investment Strategies and Cash Flows

Investment Strategy	Cash Flow at $t = 0$	Cash Flow at $t = 1$
Saving	-1	$+(1+r)$
Buying a bond (long position in bonds)	-1	$+(1+r)$

Investment Strategies and Cash Flows

Investment Strategy	Cash Flow at $t = 0$	Cash Flow at $t = 1$
Borrowing	$+1$	$-(1 + r)$
Issuing a bond	$+1$	$-(1 + r)$
Short selling a bond (short position in bonds)	$+1$	$-(1 + r)$
Selling a bond (out of inventory)	$+1$	$-(1 + r)$

Investment Strategies and Cash Flows

Investment Strategy	Cash Flow at $t = 0$	Cash Flow at $t = 1$
Buying a stock (long position in stocks)	$-P_0^s$	$+P_1^s$
Short selling a stock (short position in stocks)	$+P_0^s$	$-P_1^s$
Selling a stock (out of inventory)	$+P_0^s$	$-P_1^s$

Consumer Optimization: The Time Dimension

Someone who already owns bonds can "borrow" by selling a bond out of inventory. In fact, theories like Fisher's work better when applied to consumers who already own stocks and bonds.

Greg Mankiw and Stephen Zeldes, "The Consumption of Stockholders and Nonstockholders," *Journal of Finance*, 1991.

Annette Vissing-Jorgensen, "Limited Asset Market Participation and the Elasticity of Intertemporal Substitution," *Journal of Political Economy*, 2002.

Consumer Optimization: The Risk Dimension

In the 1950s and 1960s, Kenneth Arrow (US, 1921-2017, Nobel Prize 1972) and Gerard Debreu (France, 1921-2004, Nobel Prize 1983) extended consumer theory to accommodate risk and uncertainty.

To do so, they drew on earlier ideas developed by others, but added important insights of their own.

Building Blocks of Arrow-Debreu Theory

1. Fisher's (1930) intertemporal model of consumer decision-making.
2. From probability theory: uncertainty described with reference to "states of the world." (Andrey Kolmogorov, 1930s).
3. Expected utility theory (John von Neumann and Oskar Morgenstern, 1947).
4. Contingent claims – stylized financial assets – a powerful analytic device of their own invention.

Consumer Optimization: The Risk Dimension

To be more specific about the source of risk, let's suppose that there are two possible outcomes for income next year, good and bad:

$$Y_0 = \text{income today}$$

$$Y_1^G = \text{income next year in the ``good'' state}$$

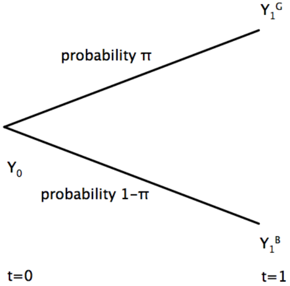
$$Y_1^B = \text{income next year in the ``bad'' state}$$

where the assumption $Y_1^G > Y_1^B$ makes the "good" state good and where

$$\pi = \text{probability of the good state}$$

$$1 - \pi = \text{probability of the bad state}$$

Consumer Optimization: The Risk Dimension



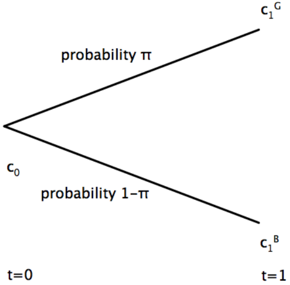
An event tree highlights randomness in income as the source of risk.

Consumer Optimization: The Risk Dimension

Arrow and Debreu used the probabilistic idea of states of the world to extend Irving Fisher's work, recognizing that under these circumstances, the consumer chooses between three goods:

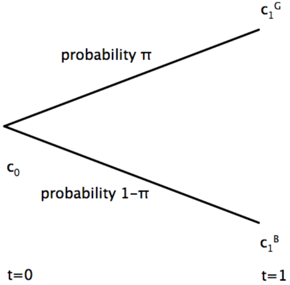
$$c_0 = \text{consumption today}$$
$$c_1^G = \text{consumption next year in the good state}$$
$$c_1^B = \text{consumption next year in the bad state}$$

Consumer Optimization: The Risk Dimension



Under uncertainty, the consumer chooses consumption today and consumption in both states next year.

Consumer Optimization: The Risk Dimension



Uncertainty about future income “induces” randomness in future consumption as well.

Consumer Optimization: The Risk Dimension

Suppose that the consumer's utility function is

$$u(c_0) + \beta\pi u(c_1^G) + \beta(1-\pi)u(c_1^B),$$

so that the terms involving next year's consumption are weighted by the probability that each state will occur as well as by the discount factor β.

Consumer Optimization: The Risk Dimension

In probability theory, if a random variable X can take on n possible values, $X_1, X_2, \ldots, X_n$, with probabilities $\pi_1, \pi_2, \ldots, \pi_n$, then the expected value of X is

$$E(X) = \pi_1 X_1 + \pi_2 X_2 + \ldots + \pi_n X_n.$$

Consumer Optimization: The Risk Dimension

Hence, by assuming that the consumer's utility function is

$$u(c_0) + \beta \pi u(c_1^G) + \beta (1 - \pi) u(c_1^B),$$

we are assuming that the consumer's seeks to maximize expected utility

$$u(c_0) + \beta E[u(c_1)].$$

Consumer Optimization: The Risk Dimension

But by writing out all three terms,

$$u(c_0) + \beta \pi u(c_1^G) + \beta(1 - \pi)u(c_1^B),$$

we can see that concavity of the function u, which in the standard microeconomic case represents a preference for diversity, represents here a preference for smoothness in consumption over time and across states in the future – the consumer is risk averse in the sense that he or she does not want consumption in the bad state to be too much different from consumption in the good state.