

ECON 337901

FINANCIAL ECONOMICS

Peter Ireland

Boston College

January 17, 2019

These lecture notes by Peter Ireland are licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0) License. <http://creativecommons.org/licenses/by-nc-sa/4.0/>.

Mathematical Preliminaries

Unconstrained Optimization

$$\max_x F(x)$$

Constrained Optimization

$$\max_x F(x) \text{ subject to } c \geq G(x)$$

Unconstrained Optimization

To find the value of x that solves

$$\max_x F(x)$$

you can:

1. Try out every possible value of x .
2. Use calculus.

Since search could take forever, let's use calculus instead.

Unconstrained Optimization

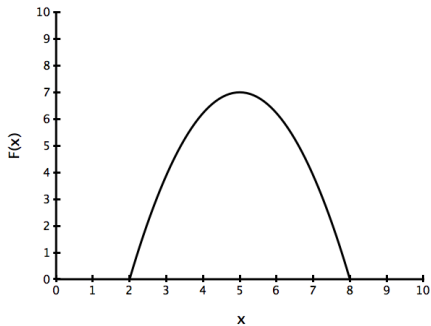
Theorem If x^* solves

$$\max_x F(x),$$

then x^* is a **critical point** of F , that is,

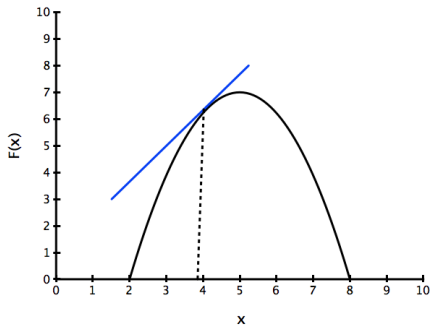
$$F'(x^*) = 0.$$

Unconstrained Optimization



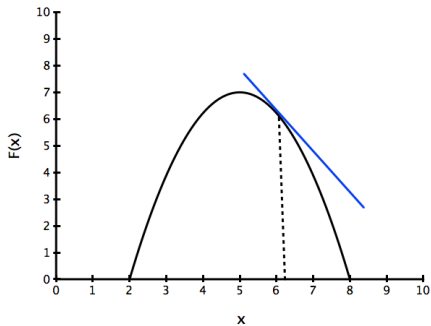
$F(x)$ maximized at $x^* = 5$

Unconstrained Optimization



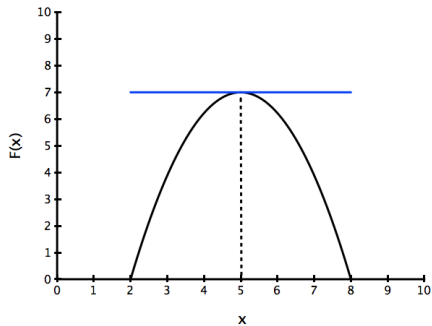
$F'(x) > 0$ when $x < 5$. $F(x)$ can be increased by increasing x .

Unconstrained Optimization



$F'(x) < 0$ when $x > 5$. $F(x)$ can be increased by decreasing x .

Unconstrained Optimization



$F'(x) = 0$ when $x = 5$. $F(x)$ is maximized.

Unconstrained Optimization

Theorem If x^* solves

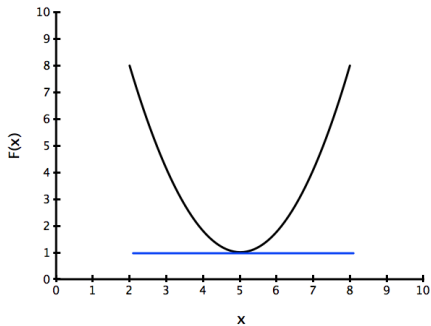
$$\max_x F(x),$$

then x^* is a **critical point** of F , that is,

$$F'(x^*) = 0.$$

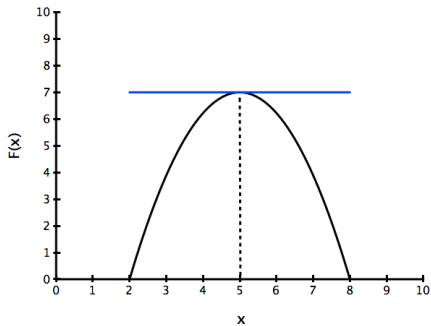
Note that the same **first-order necessary condition** $F'(x^*) = 0$ also characterizes a value of x^* that **minimizes** $F(x)$.

Unconstrained Optimization



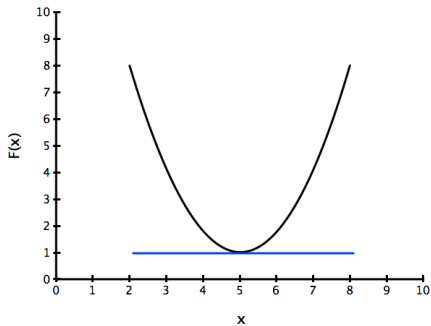
$F'(x) = 0$ when $x = 5$. $F(x)$ is minimized.

Unconstrained Optimization



$F'(x) = 0$ and $F''(x) < 0$ when $x = 5$. $F(x)$ is maximized.

Unconstrained Optimization



$F'(x) = 0$ and $F''(x) > 0$ when $x = 5$. $F(x)$ is minimized.

Unconstrained Optimization

Theorem If x^* solves

$$\max_x F(x),$$

then x^* is a **critical point** of F , that is,

$$F'(x^*) = 0.$$

Unconstrained Optimization

Theorem If

$$F'(x^*) = 0 \text{ and } F''(x^*) < 0,$$

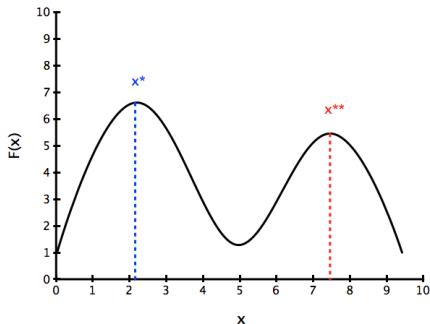
then x^* solves

$$\max_x F(x)$$

(at least locally).

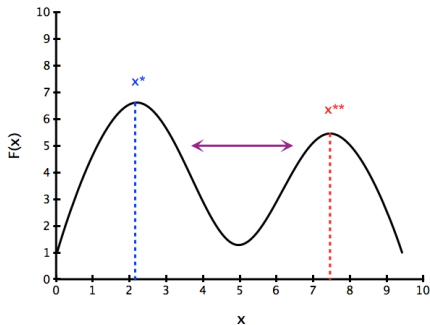
The first-order condition $F'(x^*) = 0$ and the **second-order condition** $F''(x^*) < 0$ are **sufficient** conditions for the value of x that (locally) maximizes $F(x)$.

Unconstrained Optimization



$F'(x^{**}) = 0$ and $F''(x^{**}) < 0$ at the **local** maximizer x^{**} and
 $F'(x^*) = 0$ and $F''(x^*) < 0$ at the **global** maximizer x^* .

Unconstrained Optimization



$F'(x^{**}) = 0$ and $F''(x^{**}) < 0$ at the local maximizer x^{**} and $F'(x^*) = 0$ and $F''(x^*) < 0$ at the global maximizer x^* , but $F''(x) > 0$ in between x^* and x^{**} .

Unconstrained Optimization

Theorem If

$$F'(x^*) = 0$$

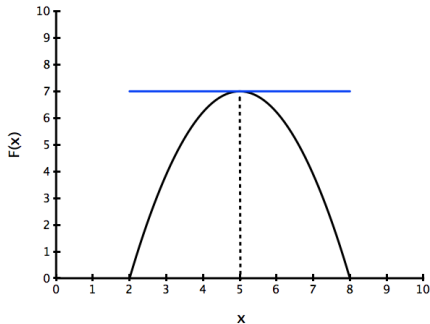
and

$$F''(x) < 0 \text{ for all } x \in \mathbb{R},$$

then x^* solves

$$\max_x F(x).$$

Unconstrained Optimization



$F''(x) < 0$ for all $x \in \mathbb{R}$ and $F'(5) = 0$. $F(x)$ is maximized when $x = 5$.

Unconstrained Optimization

If $F''(x) < 0$ for all $x \in \mathbb{R}$, then the function F is **concave**.

When F is concave, the first-order condition $F'(x^*) = 0$ is **both necessary and sufficient** for the value of x that maximizes $F(x)$.

And, as we are about to see, concave functions arise frequently and naturally in economics and finance.

Unconstrained Optimization: Example 1

Consider the problem

$$\max_x \left(-\frac{1}{2} \right) (x - \tau)^2,$$

where τ is a number ($\tau \in \mathbb{R}$) that we might call the “target.”

The first-order condition

$$-(x^* - \tau) = 0$$

leads us immediately to the solution: $x^* = \tau$.