Problem Set 9

Question 1 on problem set 9 will help us see how Arrow and Pratt's coefficient of relative risk aversion measures an investor's willingness to take on bets expressed as a fraction of income.

Questions 2 and 3, meanwhile, will allow us to appreciate how, even outside the area of financial economics, risk aversion as reflected in the concavity of the Bernoulli utility function in an expected utility framework captures risk aversion.

Interpreting the Measures of Risk Aversion

In class, we derived the approximation

$$\pi^* \approx \frac{1}{2} + \frac{1}{4} R_R(Y) k$$

for the probability π^* that makes an investor with initial income Y and coefficient of relative risk aversion $R_R(Y)$ indifferent between accepting and rejecting a bet over Yk, that is, the fraction k of income.

The example here will help us see how accurate this approximation turns out to be.

Interpreting the Measures of Risk Aversion

Suppose that the investor has initial income $Y = 10$ and is offered a bet with $k = 0.01$, that is, a bet over one percent of income.

If the investor rejects, his or her income remains at 10.

If the investor accepts, his or her income rises to 10.1 with probability π but falls to 9.9 with probability $1 - \pi$.

Interpreting the Measures of Risk Aversion

Suppose that the investor's preferences are described by a von Neumann-Morgenstern expected utility function with Bernoulii utility function

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma}$$

so that

$$R_R(Y) = -\frac{Yu''(Y)}{u'(Y)} = \gamma.$$

With $k = 0.01 = 1/100$, the approximation formula becomes

$$\pi^* \approx \frac{1}{2} + \frac{1}{4} R_R(Y) k$$

$$\pi^* \approx \frac{1}{2} + \frac{\gamma}{400}.$$

Interpreting the Measures of Risk Aversion

The exact value of π^* satisfies

$$
\begin{aligned}
u(Y) &= \pi^* u(Y + Yk) + (1 - \pi^*) u(Y - Yk) \\
u(10) &= \pi^* u(10.1) + (1 - \pi^*) u(9.9) \\
\frac{10^{1-\gamma} - 1}{1 - \gamma} &= \pi^* \left(\frac{10.1^{1-\gamma} - 1}{1 - \gamma} \right) + (1 - \pi^*) \left(\frac{9.9^{1-\gamma} - 1}{1 - \gamma} \right) \\
10^{1-\gamma} &= \pi^* (10.1^{1-\gamma}) + 9.9^{1-\gamma} - \pi^* (9.9^{1-\gamma})
\end{aligned}
$$

Interpreting the Measures of Risk Aversion

The exact value of π^* satisfies

$$10^{1-\gamma} = \pi^*(10.1^{1-\gamma}) + 9.9^{1-\gamma} - \pi^*(9.9^{1-\gamma})$$

$$\pi^* = \frac{10^{1-\gamma} - 9.9^{1-\gamma}}{10.1^{1-\gamma} - 9.9^{1-\gamma}}$$

Compute the exact values of π^* for $\gamma = 1/2, 2, 3, 10, 20$ and compare them to the approximations. They will be quite close.

Interpreting the Measures of Risk Aversion

Although it is not required for answering the problem set question, you can verify that the close accuracy of the approximation does not depend on the value of $Y = 10$.

What matters is that the size of the bet is relatively small as a fraction of income. For larger values of k, the approximation may be less accurate.

Insurance

We can also use the expected utility framework to consider a consumer's decisions about whether or not to buy insurance against a loss.

As an example, consider a consumer with income $Y = 100000$, who faces a $\pi_1 = 0.05$ probability of suffering a 50000 loss.

Suppose this consumer's preferences are described by a von Neumann-Morgenstern expected utility function with logarithmic Bernoulli utility function $u(Y) = \ln(Y)$. We know this implies a coefficient of relative risk aversion equal to one.

Insurance

What is the most this consumer will be willing to pay for insurance?

Let x^* denote the insurance premium that makes the consumer indifferent between buying and not buying insurance.

If the actual insurance premium is less than x^*, the consumer will buy the insurance; if the premium is greater than x^*, the consumer will choose to remain uninsured.

Insurance

x^* satisfies

$$u(Y - x^*) = \pi_1 u(Y - 50000) + (1 - \pi_1) u(Y)$$

$$\ln(100000 - x^*) = 0.05 \ln(50000) + 0.95 \ln(100000) \approx 11.48$$

$$\exp[\ln(100000 - x^*)] = \exp[0.05 \ln(50000) + 0.95 \ln(100000)]$$

$$100000 - x^* = \exp[0.05 \ln(50000) + 0.95 \ln(100000)]$$

$$x^* = 100000 - \exp[0.05 \ln(50000) + 0.95 \ln(100000)]$$

Just remember: $e^{y+z} \neq e^y + e^z$

Insurance

Calculate x^* again when there is

$\pi_1 = 0.05$ probability of a 50000 loss

$\pi_2 = 0.01$ probability of a 99999 loss

$1 - \pi_1 - \pi_2 = 0.94$ probability of no loss

x^* satisfies

$$u(Y - x^*) = \pi_1 u(Y - 50000) + \pi_2 u(Y - 99999) + (1 - \pi_1 - \pi_2) u(Y)$$

$$\ln(100000 - x^*) = 0.05 \ln(50000) + 0.01 \ln(1) + 0.94 \ln(100000)$$

The consumer will be willing to pay (a lot) more for insurance in this case!