Problem Set 9

ECON 337901 - Financial Economics
Boston College, Department of Economics

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For Extra Practice - Not Collected or Graded

1. Interpreting Measures of Risk Aversion

Consider an investor who has von Neumann-Morgenstern expected utility with Bernoulli utility function

$$u(Y) = \frac{Y^{1-\gamma}-1}{1-\gamma},$$

where, as we now know, $\gamma > 0$ represents the coefficient of relative risk aversion.

Consider an investor with initial wealth $Y = 10$ who is offered a bet over one percent of income. With $k = 0.01$, this means win $Yk = 0.1$ with probability π and lose $Yk = 0.1$ with probability $1-\pi$. Let π^* denote the value of π that makes the investor indifferent between accepting and rejecting this bet, and recall from class that this value of π^* can be approximated by

$$\pi^* \approx \frac{1}{2} + \frac{1}{4}\left[-\frac{Yu''(Y)}{u'(Y)}\right]k = \frac{1}{2} + \frac{1}{4}R_R(Y)k.$$

Suppose that this investor has von Neumann-Morgenstern expected utility with Bernoulli utility function

$$u(Y) = \frac{Y^{1-\gamma}-1}{1-\gamma}.$$

This utility function implies that the coefficient of relative risk aversion is constant (independent of initial income Y and equal to $R_R(Y) = \gamma$. Therefore, for this utility function and with $k = 0.01 = 1/100$, the approximation for π^* becomes, more simply,

$$\pi^* \approx \frac{1}{2} + \frac{\gamma}{400}$$

Use this formula to compute the approximate value of π^* for values of γ equal to 1/2, 2, 3, 10, and 20.

Now, let's check how accurate this approximation is by computing the exact values of π^* for each of the same five values of γ. To do this, recall that the exact value of π^* is determined by

$$u(Y) = \pi^* u(Y+Yk) + (1-\pi^*)u(Y-Yk),$$

which equates utility from rejecting the bet to expected utility from taking the bet. With $Y = 10$, $k = 0.01$, and the specific form for the utility function, this condition becomes, more specifically

$$\frac{10^{1-\gamma}-1}{1-\gamma} = \pi^*\left(\frac{10.1^{1-\gamma}-1}{1-\gamma}\right) + (1-\pi^*)\left(\frac{9.9^{1-\gamma}-1}{1-\gamma}\right).$$

This equation looks daunting at first, but notice that after multiplying both sides by $1-\gamma$, the fractions go away, leaving

$$10^{1-\gamma} - 1 = \pi^*(10.1^{1-\gamma} - 1) + (1-\pi^*)(9.9^{1-\gamma} - 1).$$

Then, adding one to both sizes simplifies the equation further:

$$10^{1-\gamma} = \pi^*(10.1^{1-\gamma}) + (1-\pi^*)(9.9^{1-\gamma}).$$

Now, in fact, the equation is simple enough to provide the solution

$$\pi^* = \frac{10^{1-\gamma} - 9.9^{1-\gamma}}{10.1^{1-\gamma} - 9.9^{1-\gamma}}.$$

Use this last formula to compute the exact value of π^* for values of γ equal to $1/2$, 2, 3, 10, and 20.

Finally, compare the exact and approximate values of π^*. You should see that, in this example at least, the approximations are very accurate.

2. Insurance, Part I

Suppose that you own a business worth \$100000. With probability $\pi_1 = 0.05$, a disaster – a fire, let's say – occurs that reduces the value of the business to \$50000. Let x denote the premium on an insurance policy that will protect you fully against that loss.

Your choices are as follows. You can take out the insurance policy, in which case your wealth will be $\$(100000 - x)$ no matter what: you'll have to pay the premium of x up front, but the insurance company will pay you \$50000 to compensate for the loss if it occurs. Or you can forego buying insurance and take your chances, in which case your wealth will be \$100000 with probability 0.95 and \$50000 with probability 0.05.

Assuming that your preferences are described by a vN-M expected utility function with logarithmic Bernoulli utility function

$$u(Y) = \ln(Y),$$

what is the maximum premium x that you will be willing to pay for the insurance policy? (*Note:* To solve this problem, you may need to use the fact that if $x = \ln(y)$, then $y = \exp(x)$. That is, the exponential function is the inverse function of the natural logarithm.

3. Insurance, Part II

Extending the example from question 2, above, suppose that in addition to the 5 percent chance of fire, there is an even smaller probability of an even bigger disaster: a flood, let's say, that occurs with probability $\pi_2 = 0.01$ but reduces the value of your business to \$1.

Your choices are now as follows. You can take out the insurance policy, in which case your wealth will be $\$(100000 - x)$ no matter what. Or you can forgo buying insurance, in which case your wealth will be \$100000 with probability 0.94, \$50000 with probability 0.05, and \$1 with probability 0.01.

Still assuming you have vN-M expected utility with logarithmic Bernoulli utility function, what is the maximum premium x that you will be willing to pay for insurance now?