Problem Set 6

In class, we derived general formulas linking stock and bond prices to contingent claim prices and then to option prices.

In problem set 6, you can use the same approach to price contingent claims and options in a special case where the "data" on the stock and bond prices and payoffs are specific numbers. Then you can check your answers by plugging the same data into the formulas derived in class.

Problem Set 6, Question 1

A stock sells for $q^s = 1.10$ today ($t = 0$) and pays a large dividend $d^G = 2$ in a good state the occurs with probability $\pi = 1/2$ and a smaller dividend $d^B = 1$ in a bad state that occurs with probability $1 - \pi = 1/2$ next year ($t = 1$).

A bond sells for $q^b = 0.90$ today and pays off one for sure (in both states) next year.

Problem Set 6, Question 1

A contingent claim for the good state pays off 1 in the good state and 0 in the bad state next year.

We want to replicate these payoffs with a portfolio consisting of s shares of stock and b bonds.

Problem Set 6, Question 1

Payoff	Claim	Portfolio
Good State	1	$d^G s + b = 2s + b$
Bad State	0	$d^B s + b = s + b$

We have a system of 2 linear equations in 2 unknowns:

$$1 = 2s + b$$

$$0 = s + b$$

Problem Set 6, Question 1

We have a system of 2 linear equations in 2 unknowns:

$$1 = 2s + b$$
$$0 = s + b$$

Use elimination (or substitution) to find the numerical values of s and b that solve these two equations.

Problem Set 6, Question 1

No arbitrage across markets for stocks, bonds, and contingent claims then requires

$$q^G = q^s s + q^b b = 1.10s + 0.90b$$

Use your numerical solutions for s and b to find the numerical value of q^G. Check your answer using the general formula from class:

$$q^G = \frac{q^s - q^b d^B}{d^G - d^B}$$

Problem Set 6, Question 1

A contingent claim for the bad state pays off 0 in the good state and 1 in the bad state next year.

We want to replicate these payoffs with a portfolio consisting of s shares of stock and b bonds.

Problem Set 6, Question 1

Payoff	Claim	Portfolio
Good State	0	$d^G s + b = 2s + b$
Bad State	1	$d^B s + b = s + b$

We have a system of 2 linear equations in 2 unknowns:

$$0 = 2s + b$$
$$1 = s + b$$

Problem Set 6, Question 1

We have a system of 2 linear equations in 2 unknowns:

$$
\begin{aligned}
0 &= 2s + b \\
1 &= s + b
\end{aligned}
$$

Use elimination (or substitution) to find the numerical values of s and b that solve these two equations.

Problem Set 6, Question 1

No arbitrage across markets for stocks, bonds, and contingent claims then requires

$$q^B = q^s s + q^b b = 1.10s + 0.90b$$

Use your numerical solutions for s and b to find the numerical value of q^B. Check your answer using the general formula from class:

$$q^B = \frac{q^b d^G - q^s}{d^G - d^B}$$

Problem Set 6, Question 2

Reinterpret the stock's dividends $d^G = 2$ and $d^B = 1$ as the prices at which the stock trades in the good and bad states next year: $P^G = 2$ and $P^B = 1$.

A call option gives the holder the right, but not the obligation, to buy a share of stock next year at the prespecified strike price K.

Problem Set 6, Question 2

Assuming that $P^G = 2 > K$, the holder will find it optimal to exercise the call when it is "in the money" in the good state, and thereby receive the payoff $2 - K$.

Assuming that $K > P^B = 1$, the holder will find it optimal not to exercise the call when it is "out of the money" in the bad state, and thereby receive the payoff of zero.

Following Robert Merton, we want to replicate these payoffs with a portfolio consisting of s shares of stock and b bonds.

Problem Set 6, Question 2

Payoff	Option	Portfolio
Good State	$2 - K$	$P^G s + b = 2s + b$
Bad State	0	$P^B s + b = s + b$

We have a system of 2 linear equations in 2 unknowns

$$2 - K = 2s + b$$
$$0 = s + b$$

that we can use to find the solutions for s and b in terms of K.

Problem Set 6, Question 2

No arbitrage across markets for stocks, bonds, and options then requires

$$q^o = q^s s + q^b b = 1.10s + 0.90b$$

Should q^o rise or fall when K increases?

Use your numerical solutions for s and b to find the numerical value of q^o. Check your answer using the general formula from class:

$$q^o = \frac{(q^s - q^b P^B)(P^G - K)}{P^G - P^B}$$

Problem Set 6, Question 2

Assuming that $P^G = 2 > K$, the holder will find it optimal to exercise the call when it is "in the money" in the good state, and thereby receive the payoff $2 - K$.

Assuming that $K > P^B = 1$, the holder will find it optimal not to exercise the call when it is "out of the money" in the bad state, and thereby receive the payoff of zero.

Finally, notice that the payoffs from the option can also be replicated by buying $2 - K$ contingent claims for the good state.

Problem Set 6, Question 2

Finally, notice that the payoffs from the option can also be replicated by buying $2 - K$ contingent claims for the good state.

No arbitrage across the markets for options and contingent claims therefore requires

$$q^o = q^G(2 - K).$$

You can also use your numerical solution for q^G from question 1 to check your answer for question 2.