

## Problem Set 6

ECON 337901 - Financial Economics  
Boston College, Department of Economics

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### 1. Stocks, Bonds, and Contingent Claims

Consider an economic environment with risk, similar to one we studied in class, in which there are two periods,  $t = 0$  and  $t = 1$ , and two possible states at  $t = 1$ : a “good” state that occurs with probability  $\pi = 1/2$  and a “bad” state that occurs with probability  $1 - \pi = 1/2$ . Suppose that investors trade two assets in this economy. The first is a “stock,” which sells for  $q^s = 1.1$  at  $t = 0$  and pays a large dividend of  $d^G = 2$  in the good state at  $t = 1$  and a small dividend of  $d^B = 1$  in the bad state at  $t = 1$ . The second is a “bond,” which sells for  $q^b = 0.9$  at  $t = 0$  and makes a payoff of one for sure, in both states, at  $t = 1$ .

Suppose, as we did in class, that investors can take long and short positions in both assets. Given the payoffs for the stock and bond, find the combination of stock and bond holdings  $s$  and  $b$  that replicate the payoff provided by a contingent claim for the good state: one in good state and zero in the bad. Then find the combination of stock and bond holdings that replicate the payoff provided by the contingent claim for the bad state. Finally, given the answers you just derived and the prices of the stock and bond, find the prices at which each contingent claim should sell at  $t = 0$ .

### 2. Option Pricing

In 1973, Fischer Black and Myron Scholes published a paper titled “The Pricing of Options and Corporate Liabilities” in the *Journal of Political Economy*, in which they solved a difficult mathematical problem involving a “stochastic differential equation” in order to derive what today in their honor is known as the “Black-Scholes” option-pricing formula. At the same time that Black and Scholes were writing their paper, another economist, Robert Merton, pointed out that the same formula could be derived more easily using Arrow and Debreu’s idea of replicating payoffs on complex assets using portfolios of contingent claims. Although Merton applied his argument to the more detailed model, with multiple time periods and many future states, originally used by Black and Scholes, his insight is so basic and powerful that it can be illustrated using the same example that you just studied above.

To apply the example from question 1 to option pricing, it is helpful to start by reinterpreting the dividends  $d^G = 2$  and  $d^B = 1$  paid by the stock in the good and bad state as the prices at which the stock trades in the same two states. Modifying the notation to reflect this new interpretation, let  $P^G = 2$  and  $P^B = 1$  be the price of the stock in the good and bad states at  $t = 1$ . Thus, the payoffs from holding the stock are the same as in question 1, but those payoffs come in the form of capital gains and losses instead of dividends.

A *stock option* is a contract that gives the holder the right, but not the obligation, to purchase

one share of the stock at  $t = 1$  at a price, called the “strike price” and usually denoted by  $K$ , that is agreed upon in advance at  $t = 0$ . To make this example interesting, it’s helpful to assume that the strike price for the option we are considering lies somewhere between 1 and 2, the prices  $P^B$  and  $P^G$  at which the stock can trade at  $t = 1$ . Then, because the owner of the stock option has the right, but not the obligation, to buy the shares at the strike price at  $t = 1$ , we can easily characterize his or her optimal decision as to whether or not to “exercise” the option. In particular, in the good state, the option-holder can buy the stock for  $K < 2$ , immediately sell it for  $P^G = 2$ , and thereby earn the positive profit  $2 - K > 0$ . It therefore makes sense for the option-holder to exercise the option in the good state. On the other hand, in the bad state, the option-holder would lose money by exercising the option and paying  $K > 1$  for the stock when all that he or she could get for the shares in the market is  $P^B = 1$ . It therefore makes sense for the option-holder to allow the option to expire, unexercised, in the bad state.

Based on these considerations, we now know the payoffs from a stock option with strike price  $K$  satisfying  $1 < K < 2$ . They are  $2 - K$  in the good state and zero in the bad state.

Merton’s original idea actually side-stepped the use of contingent claims and replicated the payoffs from the option using a portfolio of the stock and bond. We can do this here, as well. Let  $s$  denote the number of shares and  $b$  the number of bonds needed to form the portfolio that replicates the payoffs from the option. In the good state, we want this portfolio to have payoff  $2 - K$ . Remembering that the stock sells for  $P^G = 2$  in the good state at  $t = 1$  and the bond pays off one, no matter what, at  $t = 1$ , this condition requires that  $s$  and  $b$  satisfy

$$2 - K = 2s + b.$$

In the bad state, we want the portfolio to have payoff zero. Remembering that the stock sells for  $P^B = 1$  in the bad state at  $t = 1$  and the bond pays off one for sure, this condition requires that  $s$  and  $b$  satisfy

$$0 = s + b.$$

Use these two equations to find the number of shares  $s$  and bonds  $b$  that are needed to form the portfolio that replicates the payoffs on the stock option. When doing this, keep in mind this portfolio may require a short position in one of the assets. Then, using the fact that the stock sells for  $q^s = 1.1$  and the bond for  $q^b = 0.9$  at  $t = 0$ , compute the cost of assembling this portfolio. If there are no arbitrage opportunities across the markets for stocks, bonds, and options, then the cost of the portfolio will equal the price of the option. What does your answer say about the relationship between the option price and the strike price  $K$ ? When  $K$  is higher, is the option price higher or lower?

An even simpler approach to option pricing than the one originally suggested by Merton is to view the option as a portfolio of contingent claims instead of the stock and bond. From above, we saw that the option pays off  $2 - K$  in the good state and zero in the bad. What portfolio of contingent claims replicates these payoffs? What happens if you use the contingent claims prices you found in answering question 1 to compute the cost of this portfolio of contingent claims? The implied price for the option should be exactly the same as the one you derived above, using Merton’s approach . . . try it out and see!