

Problem Set 5

ECON 337901 - Financial Economics
Boston College, Department of Economics

Peter Ireland
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1. Consumer Optimization Under Uncertainty

In class, we characterized the optimizing behavior of a consumer making choices under uncertainty in an environment with two periods, today ($t = 0$) and next year ($t = 1$), and two states, good and bad, in period $t = 1$. Letting c_0 denote consumption today and c_1^G and c_1^B denote consumption in the good and bad states next year, we assumed that the consumer's utility could be written as

$$u(c_0) + \beta\pi u(c_1^G) + \beta(1 - \pi)u(c_1^B),$$

where β is a discount factor that captures the consumer's degree of patience or impatience and π denotes the probability that the good state occurs next year. Similarly, letting Y_0 , Y_1^G , and Y_1^B denote the consumer's income today and in the good and bad states next year, we wrote the consumer's budget constraint as

$$Y_0 + q^G Y_1^G + q^B Y_1^B \geq c_0 + q^G c_1^G + q^B c_1^B,$$

where q^G is the price of a contingent claim for the good state next year and q^B is the price of a contingent claim for the bad state next year.

Consider a special case of this problem, in which $\beta = 3/4$, $\pi = 1/2$, $1 - \pi = 1/2$, $Y_0 = 90$, $Y_1^G = 160$, $Y_1^B = 20$, $q^G = 1/4$, $q^B = 1/2$, and the utility function $u(c) = \ln(c)$ takes the natural log form, so that $u'(c) = 1/c$. With these settings, the consumer's problem becomes, more specifically, one of choosing c_0 , c_1^G , and c_1^B to maximize expected utility

$$\ln(c_0) + \left(\frac{3}{8}\right) \ln(c_1^G) + \left(\frac{3}{8}\right) \ln(c_1^B)$$

subject to the budget constraint

$$140 \geq c_0 + \left(\frac{1}{4}\right) c_1^G + \left(\frac{1}{2}\right) c_1^B.$$

After setting up the Lagrangian

$$L = \ln(c_0) + \left(\frac{3}{8}\right) \ln(c_1^G) + \left(\frac{3}{8}\right) \ln(c_1^B) + \lambda \left[140 - c_0 - \left(\frac{1}{4}\right) c_1^G - \left(\frac{1}{2}\right) c_1^B \right]$$

for this problem, differentiate L by each of the three choice variables and set each of the results equal to zero to obtain the three first-order conditions for the optimal values of c_0^* ,

c_1^{G*} , and c_1^{B*} . Since these first-order conditions will also make reference to the value of the Lagrange multiplier λ^* associated with the solution to the consumer's problem, you'll need to combine the three first-order order conditions with the binding budget constraint,

$$140 = c_0^* + \left(\frac{1}{4}\right) c_1^{G*} + \left(\frac{1}{2}\right) c_1^{B*},$$

to obtain a system of four equations in all four unknowns.

Although there are many ways of solving this four-equation system, probably the easiest is to follow the same steps you took in answering the question from the second problem set: (1) Use the first-order conditions to find c_0^* , c_1^{G*} , and c_1^{B*} in terms of λ^* , (2) substitute these expressions into the budget constraint to solve for the numerical value of λ^* , and (3) substitute this solution for λ^* back into the earlier expressions for c_0^* , c_1^{G*} , and c_1^{B*} to find numerical solutions for the three optimally-chosen consumptions.

Once you have your solutions for c_0^* , c_1^{G*} , and c_1^{B*} , compare them to the original values for income $Y_0 = 90$, $Y_1^G = 160$, and $Y_1^B = 20$. Although in this example, the consumer does not succeed in completely smoothing his or her consumption out over time and across the two future states, you should see that the concavity of the logarithmic utility function and the preference for smoothness in consumption it implies does work to make consumption less volatile than income.