1. Consumer Optimization

The figure below illustrates the familiar tangency condition between an optimizing consumer’s indifference curve and budget constraint. In the graph, bundle C is infeasible (unaffordable), bundle A is suboptimal, and bundle B is optimal.

Let \( c^*_a \) and \( c^*_b \) denote the number of apples and bananas included in the optimal bundle B. From the graph, there are two conditions that these quantities should satisfy.

First, \( c^*_a \) and \( c^*_b \) are on the consumer’s budget constraint. This means that

\[
c^*_b = \frac{Y}{p_b} - \left( \frac{p_a}{p_b} \right) c^*_a,
\]

(1)

where \( Y \) is the consumer’s income and \( p_a \) and \( p_b \) are the prices of apples and bananas.

Second, the slope of the indifference curve equations the slope of the budget constraint. The slope of the budget constraint is \(- (p_a/p_b)\). Suppose that consumer’s utility function takes
the form
\[ \alpha u(c_a) + (1 - \alpha)u(c_b), \]
where \( \alpha \) satisfies \( 0 < \alpha < 1 \) and measures how much the consumer likes apples relative to bananas. With this utility function, the slope of the indifference curve equals the marginal rate of substitution
\[ -\frac{\alpha u'(c_a^*)}{(1 - \alpha)u'(c_b^*)}. \]
If the consumer prefers more to less and has a preference for variety, then the function \( u(c) \) that measures utility from each of the two goods separately should be increasing and concave. A convenient choice that has both of these properties is the natural log function \( u(c) = \ln(c) \). With this function, the marginal rate of substitution becomes, more simply,
\[ -\frac{\alpha c_b^*}{(1 - \alpha)c_a^*}. \]
This last expression lets us write the condition summarizing the equality of the slopes of the indifference curve and budget constraint as
\[ -\frac{p_a}{p_b} = -\frac{\alpha c_b^*}{(1 - \alpha)c_a^*}, \]
or, canceling the minus signs and cross-multiplying to get rid of the fractions,
\[ (1 - \alpha)p_a c_a^* = \alpha p_b c_b^*. \]

Equations (1) and (2) form a system of two equations that can be used to solve for the optimal consumptions \( c_a^* \) and \( c_b^* \) in terms of income \( Y \), the prices \( p_a \) and \( p_b \), and the weight \( \alpha \) from the utility function. Although there are many ways of doing this, probably the easiest is to substitute the expression for \( c_b^* \) given by equation (1) into the right-hand side of equation (2), and solve algebraically for \( c_a^* \). Then, you can substitute this solution for \( c_a^* \) back into (1) to find the solution for \( c_b^* \). Try it out and see. Because the special utility function here is the same as the one from question 2 on the first problem set, your answers should turn out to be the same, too.