Problem Set 1

Problem set 1 presents two economic examples that will give you practice solving unconstrained optimization problems.

Use the math to derive the solutions; then use your economics as a "cross check."

Both examples show why concavity of the objective function is often a natural assumption to make in economics.

Question 1

A firm produces y units of output with n workers according to the production function

$$y = n^{\alpha}$$

where $0 < \alpha < 1$.

Fractional exponent? What does this mean? What does this imply?

Question 1

Recall that

$$y = x^{1/2} \text{ means } y = \sqrt{x}$$

Now consider

$$y = x^{a/b} \Rightarrow y^b = (x^{a/b})^b = x^a$$

In words: $y = x^{a/b}$ is the number that when raised to the power b equals x raised to the power a.

Therefore

$$y = x^{1/2} \text{ means } y^2 = x$$

Question 1

Now consider a negative exponent

$$y = x^{-a}$$

What does this mean and imply?

A negative exponent becomes positive when flipped into the denominator of a fraction:

$$y = x^{-a} \Rightarrow y = \frac{1}{x^a}$$

Question 1

A negative exponent becomes positive when flipped into the denominator of a fraction:

$$y = x^{-a} \Rightarrow y = \frac{1}{x^a}$$

It's tempting to say that if x is a positive number, $y = x^{-a}$ will be negative. But this is not correct!

Example:

$$y = 2^{-2} = \frac{1}{2^2} = \frac{1}{4} > 0.$$

Question 1

Now let's return to the production function from our example

$$y = n^{\alpha}$$

with $0 < \alpha < 1$

$$\frac{dy}{dn} = \underset{+}{\alpha} \underset{+}{n^{\alpha-1}} > 0$$

So $\alpha > 0$ implies that the marginal product of labor (MPL) is always positive.

Question 1

$$y = n^{\alpha}$$

$$\frac{dy}{dn} = \alpha n^{\alpha-1} > 0$$

$$\frac{d^2y}{dn^2} = (\underset{-}{\alpha - 1})\underset{+}{\alpha}\underset{+}{n^{\alpha-2}} < 0$$

So $\alpha < 1$ implies that the MPL decreases as more workers are hired. Concavity is a reasonable assumption because it reflects diminishing marginal returns.

Question 1

By the way, if $y(t)$ describes y as a function of t

$$y'(t)$$

$$\frac{dy}{dt}$$

and

$$\dot{y}(t)$$

all stand for the same thing: the first derivative of y with respect of t.

The first symbol was used by Joseph-Louis Lagrange, the second by Gottfried Leibniz, and the third by Isaac Newton.

Question 1

Now let w be the wage rate and measure profits by

$$F(n) = n^{\alpha} - wn$$

Then

$$F'(n) = \alpha n^{\alpha-1} - w$$

and

$$F''(n) = (\alpha - 1)\alpha n^{\alpha-2} < 0$$

Since the profit-maximizing firm's objective function is concave, the FOC is both necessary and sufficient for a global maximum.

Question 1

The firm's problem

$$\max_{n} n^{\alpha} - wn$$

Use the FOC to find n^* in terms of w and α.

Given that $0 < \alpha < 1$, what happens to n^* when w goes up?

Hint: The solution linking n^* to w is the firm's labor demand curve, and there are no Giffen goods in producer theory!

Question 2

Reinterpret the production function from Q1 as one that links a farmer's consumption c to hours worked h:

$$c = h^{\alpha}$$

again with $0 < \alpha < 1$.

Suppose the farmer's preferences over c and h are described by the utility function

$$\ln(c) - \beta h$$

where ln is the natural logarithm and $\beta > 0$ measures the farmer's distaste for work.

Question 2

Recall that

$$\ln(x^a) = a\ln(x)$$

Recall also that

$$f(x) = \ln(x)$$

implies

$$f'(x) = \frac{1}{x} = x^{-1}$$

and

$$f''(x) = -x^{-2} = -\frac{1}{x^2}$$

Question 2

With

$$f(x) = \ln(x)$$

$$f'(x) = \frac{1}{x} = x^{-1} > 0$$

in economics: more preferred to less (positive marginal utility MU).

And

$$f''(x) = -x^{-2} = -\frac{1}{x^2} < 0$$

in economics: concavity reflects diminishing MU.

Question 2

Production:

$$c = h^{\alpha}$$

Utility:

$$\ln(c) - \beta h$$

The logarithmic utility function, though seemingly obscure, is analytically convenient and reflects economic assumptions about preferences that seem reasonable.

Substitute the production function into the utility function to express the problem as

$$\max_{h} \; \alpha \ln(h) - \beta h$$

Question 2

$$\max_{h} \ \alpha \ln(h) - \beta h$$

$$F(h) = \alpha \ln(h) - \beta h$$
$$F'(h) = \frac{\alpha}{h} - \beta = \alpha h^{-1} - \beta$$
$$F''(h) = -\alpha h^{-2} = -\frac{\alpha}{h^2} < 0$$

Concavity of the objective function means the FOC will lead us reliably to the solution.

Question 2

$$\max_{h} \ \alpha \ln(h) - \beta h$$

Use the FOC for find h^* in terms of α and β.

It turns out that h^* increases when α rises.

What happens to h^* when β goes up?