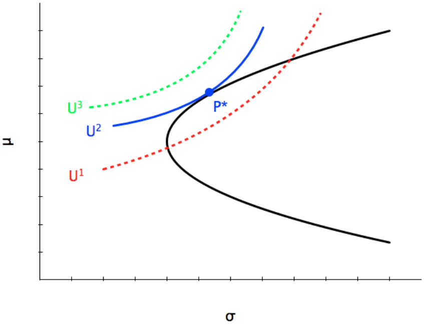
Problem Set 13

Problem Set 13 asks you to work through an algebraic analysis of optimal portfolio allocation when, following Harry Markowitz, we assume that investors have preferences that can be described by a mean-variance utility function.

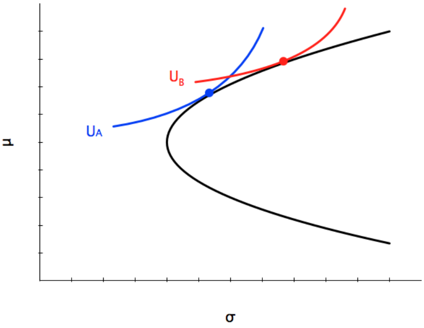
The analysis parallels the graphical analysis from class.

Problem Set 13



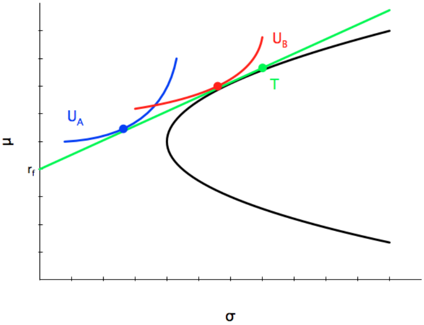
Portfolios along U^1 are suboptimal. Portfolios along U^3 are infeasible. Portfolio P^*, located where U^2 is tangent to the efficient frontier, is optimal.

Problem Set 13



Investor B is less risk averse than investor A. But both choose portfolios along the efficient frontier.

Problem Set 13



Investor B is less risk averse than investor A. But both choose some combination of the "tangency portfolio" T and the risk free asset.

Problem Set 13, Questions 1 and 2

Suppose the investor's utility function is

$$U(\mu_p, \sigma_p^2) = \mu_p - \left(\frac{A}{2}\right)\sigma_p^2$$

where higher values of A correspond to greater risk aversion.

Again, note: this is a mean-variance utility function, not an expected utility function.

Problem Set 13, Questions 1 and 2

If this investor chooses between a risky portfolio (perhaps the tangency portfolio) and a risk-free asset

$$\mu_p = r_f + \left(\frac{\mu_r - r_f}{\sigma_r}\right)\sigma_p$$

and

$$\begin{aligned} U(\mu_p, \sigma_p^2) &= \mu_p - \left(\frac{A}{2}\right)\sigma_p^2 \\ &= r_f + \left(\frac{\mu_r - r_f}{\sigma_r}\right)\sigma_p - \left(\frac{A}{2}\right)\sigma_p^2 \end{aligned}$$

Problem Set 13, Questions 1 and 2

The investor's problem is

$$\max_{\sigma_p} \; r_f + \left(\frac{\mu_r - r_f}{\sigma_r} \right) \sigma_p - \left(\frac{A}{2} \right) \sigma_p^2$$

Problem Set 13, Questions 1 and 2

The investor's problem

$$\max_{\sigma_p} \; r_f + \left(\frac{\mu_r - r_f}{\sigma_r}\right)\sigma_p - \left(\frac{A}{2}\right)\sigma_p^2$$

has FOC

$$\frac{\mu_r - r_f}{\sigma_r} - A\sigma_p^* = 0$$

Use the FOC to find σ_p^* in terms of $\mu_r - r_f$, σ_r, and A. Then use

$$w^* = \frac{\sigma_p^*}{\sigma_r}.$$

to see how w^* depends on $\mu_r - r_f$, σ_r, and A.

Problem Set 13, Questions 1 and 2

w^*, the share optimally allocated to the risky portfolio:

1. Rises when $\mu_r - r_f$ increases.
2. Falls when σ_r or A increases.

Echoing the results from Arrow (1971), which assume expected utility instead.

Problem Set 13, Question 3

Extension with 2 risky assets:

random returns $\tilde{r}_1$, $\tilde{r}_2$

expected returns μ_1, μ_2

standard deviations σ_1, σ_2

uncorrelated returns $\rho_{12} = 0$

shares w_1, w_2 allocated to these two risky assets

remaining share $1 - w_1 - w_2$ allocated to the risk-free asset

Problem Set 13, Question 3

The utility function is still

$$U(\mu_p, \sigma_p^2) = \mu_p - \left(\frac{A}{2}\right)\sigma_p^2$$

But the mean and variance of the portfolio are

$$\mu_p = (1 - w_1 - w_2)r_f + w_1\mu_1 + w_2\mu_2$$

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2$$

Problem Set 13, Question 3

The investor's problem is

$$\max_{w_1, w_2} \; \mu_p - \left(\frac{A}{2}\right) \sigma_p^2$$

$$\max_{w_1, w_2} \; (1 - w_1 - w_2) r_f + w_1 \mu_1 + w_2 \mu_2 - \left(\frac{A}{2}\right) \left(w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2\right)$$

Problem Set 13, Question 3

The investor's problem

$$\max_{w_1, w_2} (1 - w_1 - w_2)r_f + w_1\mu_1 + w_2\mu_2 - \left(\frac{A}{2}\right)(w_1^2\sigma_1^2 + w_2^2\sigma_2^2)$$

has FOCs:

$$-r_f + \mu_1 - A\sigma_1^2 w_1^* = 0$$
$$-r_f + \mu_2 - A\sigma_2^2 w_2^* = 0$$

Use these FOCs to see how w_1^* and w_2^* depend on $\mu_1 - r_f$, $\mu_2 - r_f$, σ_1, σ_2, and A.

Problem Set 13, Question 3

Although w_1^* and w_2^* both decrease when risk aversion rises, the ratio w_1^*/w_2^* is independent of A.

This result is also implied by the MPT's two-fund or separation theorem, which says that all investors will hold the same portfolio of risky assets, namely the tangency portfolio, and scale back on risk by substituting into risk-free assets instead.