

## Problem Set 13

ECON 337901 - Financial Economics  
Boston College, Department of Economics

Peter Ireland  
Spring 2020

For Extra Practice - Not Collected or Graded

### 1. Portfolio Allocation with Mean-Variance Utility, Part I

Consider an investor with preferences over the mean and variance of the returns on his or her portfolio that are described by the utility function

$$U(\mu_P, \sigma_P^2) = \mu_P - \left(\frac{A}{2}\right) \sigma_P^2,$$

where a higher value of  $A$  corresponds to a larger aversion to risk. Suppose that this investor is able to form a portfolio from a risk-free asset with return  $r_f$  and a risky asset with expected return equal to  $\mu_r$  and a standard deviation of its return equal to  $\sigma_r$ . In this context, the risky asset may simply be the only risky asset available to the investor or it can be the tangency portfolio that represents the optimal combination of many individual risky assets. Either way, our analysis from class showed that the relationship between  $\mu_P$  and  $\sigma_P$  for the portfolio that combines the risk-free asset with the risky asset is linear:

$$\mu_P = r_f + \left(\frac{\mu_r - r_f}{\sigma_r}\right) \sigma_P.$$

By substituting this constraint into the investor's utility function, his or her problem can be solved as one of choosing  $\sigma_P$  to maximize

$$r_f + \left(\frac{\mu_r - r_f}{\sigma_r}\right) \sigma_P - \left(\frac{A}{2}\right) \sigma_P^2.$$

Use the first-order condition for this problem to solve for the optimal choice of  $\sigma_P$ .

### 2. Portfolio Allocation with Mean-Variance Utility, Part II

Our analysis from class also showed that the portfolio formed from the risk-free and risky asset that has a return with standard deviation  $\sigma_P$  allocates the fraction

$$w = \frac{\sigma_P}{\sigma_r}$$

of wealth to the risk asset and the remaining fraction  $1 - w$  to the risk-free asset. Use this expression, together with your solution for the optimal choice of  $\sigma_P$  from question 3, above, to find the investors' optimal choice of  $w$ . How does the optimal share of the risky asset depend on the investor's risk aversion, as measured by the parameter  $A$  from the utility function? How does the optimal share of the risky asset depend on  $\mu_r - r_f$ , the risk premium defined as the expected return on the risky asset minus the risk-free return?

### 3. Portfolio Allocation with Mean-Variance Utility, Part III

Suppose the same investor that you considered above, with mean-variance utility function

$$U(\mu_P, \sigma_P^2) = \mu_P - \left(\frac{A}{2}\right) \sigma_P^2,$$

can now form his or her portfolio from two risky assets, one with random return  $\tilde{r}_1$  having expected value  $\mu_1 = E(\tilde{r}_1)$  and standard deviation  $\sigma_1$  and the second with random return  $\tilde{r}_2$  having expected value  $\mu_2 = E(\tilde{r}_2)$  and standard deviation  $\sigma_2$ , as well as the risk-free asset with return  $r_f$ . Assuming for simplicity that the returns on the two risky assets are uncorrelated, a portfolio with share  $w_1$  allocated to risky asset 1, share  $w_2$  allocated to risky asset 2, and the remaining share  $1 - w_1 - w_2$  allocated to the risk-free asset has return with expected value

$$\mu_P = (1 - w_1 - w_2)r_f + w_1\mu_1 + w_2\mu_2$$

and variance

$$\sigma_P^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2.$$

By substituting these expressions for  $\mu_P$  and  $\sigma_P^2$  into the utility function, the investor's problem can be described as one of choosing  $w_1$  and  $w_2$  to maximize

$$(1 - w_1 - w_2)r_f + w_1\mu_1 + w_2\mu_2 - \left(\frac{A}{2}\right) (w_1^2\sigma_1^2 + w_2^2\sigma_2^2).$$

Use the first-order conditions for the investor's optimal choices  $w_1^*$  and  $w_2^*$  to see how these optimal portfolio shares depend on the investor's risk aversion, as measured by the parameter  $A$  from the utility function, the risk premium  $\mu_1 - r_f$  and  $\mu_2 - r_f$  for each risky asset, and the volatility as measured by the variance  $\sigma_1^2$  and  $\sigma_2^2$  of each random return.

### 4. CAPM Betas and Expected Returns

Here are some stocks with high and low betas, as reported by the *Value Line Investment Survey*:

High Beta Stocks		Low Beta Stocks	
Company	Beta	Company	Beta
Intel	1.10	Kellogg	0.60
Ford Motor	1.15	Procter & Gamble	0.65
Caterpillar	1.35	Wal-Mart	0.75
US Steel	2.20	Pfizer	0.95

Suppose that the risk-free rate is  $r_f = 0.02$  (2 percent) and that the expected return on the market portfolio is  $E(\tilde{r}_M) = 0.08$  (8 percent), then use the CAPM to calculate the expected returns on each of these eight stocks.