Problem Set 13

ECON 337901 - Financial Economics — Peter Ireland
Boston College, Department of Economics — Fall 2024

For Extra Practice - Not Collected or Graded

1. Portfolio Allocation with Mean-Variance Utility, Part I

Consider an investor with preferences over the mean and variance of the returns on his or her portfolio that are described by the utility function

$$U(\mu_P, \sigma_P^2) = \mu_P - \left(\frac{A}{2}\right)\sigma_P^2,$$

where a higher value of A corresponds to a larger aversion to risk. Suppose that this investor is able to form a portfolio from a risk-free asset with return r_f and a risky asset with expected return equal to μ_r and a standard deviation of its return equal to σ_r. In this context, the risky asset may simply be the only risky asset available to the investor or it can be the tangency portfolio that represents the optimal combination of many individual risky assets. Either way, our analysis from class showed that the relationship between μ_P and σ_P for the portfolio that combines the risk-free asset with the risky asset is linear:

$$\mu_P = r_f + \left(\frac{\mu_r - r_f}{\sigma_r}\right)\sigma_P.$$

By substituting this constraint into the investor's utility function, his or her problem can be solved as one of choosing σ_P to maximize

$$r_f + \left(\frac{\mu_r - r_f}{\sigma_r}\right)\sigma_P - \left(\frac{A}{2}\right)\sigma_P^2.$$

Use the first-order condition for this problem to solve for the optimal choice of σ_P.

2. Portfolio Allocation with Mean-Variance Utility, Part II

Our analysis from class also showed that the portfolio formed from the risk-free and risky asset that has a return with standard deviation σ_P allocates the fraction

$$w = \frac{\sigma_P}{\sigma_r}$$

of wealth to the risk asset and the remaining fraction $1 - w$ to the risk-free asset. Use this expression, together with your solution for the optimal choice of σ_P from question 1, above, to find the investors' optimal choice of w. How does the optimal share of the risky asset depend on the investor's risk aversion, as measured by the parameter A from the utility function? How does the optimal share of the risky asset depend on $\mu_r - r_f$, the risk premium defined as the expected return on the risky asset minus the risk-free return?

3. Portfolio Allocation with Mean-Variance Utility, Part III

Suppose the same investor that you considered above, with mean-variance utility function

$$U(\mu_P, \sigma_P^2) = \mu_P - \left(\frac{A}{2}\right)\sigma_P^2,$$

can now form his or her portfolio from two risky assets, one with random return $\tilde{r}_1$ having expected value $\mu_1 = E(\tilde{r}_1)$ and standard deviation σ_1 and the second with random return $\tilde{r}_2$ having expected value $\mu_2 = E(\tilde{r}_2)$ and standard deviation σ_2, as well as the risk-free asset with return r_f. Assuming for simplicity that the returns on the two risky assets are uncorrelated, a portfolio with share w_1 allocated to risky asset 1, share w_2 allocated to risky asset 2, and the remaining share $1 - w_1 - w_2$ allocated to the risk-free asset has return with expected value

$$\mu_P = (1 - w_1 - w_2)r_f + w_1\mu_1 + w_2\mu_2$$

and variance

$$\sigma_P^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2.$$

By substituting these expressions for μ_P and σ_P^2 into the utility function, the investor's problem can be described as one of choosing w_1 and w_2 to maximize

$$(1 - w_1 - w_2)r_f + w_1\mu_1 + w_2\mu_2 - \left(\frac{A}{2}\right)(w_1^2\sigma_1^2 + w_2^2\sigma_2^2).$$

Use the first-order conditions for the investor's optimal choices w_1^* and w_2^* to see how these optimal portfolio shares depend on the investor's risk aversion, as measured by the parameter A from the utility function, the risk premium $\mu_1 - r_f$ and $\mu_2 - r_f$ for each risky asset, and the volatility as measured by the variance σ_1^2 and σ_2^2 of each random return.