Problem Set 12

ECON 337901 - Financial Economics — Peter Ireland
Boston College, Department of Economics — Spring 2024

Due Tuesday, April 23

1. The Efficient Frontier

In problem set 11, you considered portfolios formed from two risky assets, the first with expected return equal to $\mu_1 = 8$ and standard deviation of its return equal to $\sigma_1 = 8$ and the second with expected return equal to $\mu_2 = 4$ and standard deviation of its return equal to $\sigma_2 = 4$. Suppose now that in addition to these two risky assets, we have a third, with expected return equal to $\mu_3 = 6$ and standard deviation of its return equal to $\sigma_3 = 6$.

Obviously, a 6 percent expected return can be achieved with a standard deviation of 6 percent simply by holding this third asset. The question is how much better one can do, in terms of a lower standard deviation, by holding the optimal diversified portfolio.

To find out, suppose that all three asset returns are uncorrelated, so that $\rho_{12} = \rho_{13} = \rho_{23} = 0$. Then consider the problem: choose fractions w_1, w_2 and $1-w_1-w_2$ of a portfolio allocated to assets 1, 2, and 3 in order to maximize $-\sigma_P^2$, minus one times the variance of the portfolio's return, subject to the constraint that μ_P, the expected return on the portfolio, equals $\bar{\mu} = 6$.

Since $\mu_1 = 8$, $\mu_2 = 4$, and $\mu_3 = 6$,

$$\mu_P = \mu_1 w_1 + \mu_2 w_2 + \mu_3(1 - w_1 - w_2) = 8w_1 + 4w_2 + 6(1 - w_1 - w_2).$$

And since $\sigma_1 = 8$, $\sigma_2 = 4$, $\sigma_3 = 6$, and the asset returns are all uncorrelated,

$$\begin{aligned} \sigma_P^2 &= \sigma_1^2 w_1^2 + \sigma_2^2 w_2^2 + \sigma_3^2 (1 - w_1 - w_2)^2 \\ &= 64w_1^2 + 16w_2^2 + 36(1 - w_1 - w_2)^2 \end{aligned}$$

Hence, the problem

$$\max_{w_1, w_2} -\sigma_P^2 \text{ subject to } \mu_P = \bar{\mu}$$

can in this case be written more specifically as

$$\max_{w_1, w_2} -64w_1^2 - 16w_2^2 - 36(1 - w_1 - w_2)^2 \text{ subject to } 8w_1 + 4w_2 + 6(1 - w_1 - w_2) = 6.$$

Take the first-order conditions by differentiating the Lagrangian

$$L(w_1, w_2, \lambda) = -64w_1^2 - 16w_2^2 - 36(1 - w_1 - w_2)^2 + \lambda[8w_1 + 4w_2 + 6(1 - w_1 - w_2) - 6]$$

through first by w_1^* and then by w_2^*, in each case setting the result equal to zero. Then use these first-order conditions, together with the constraint

$$8w_1^* + 4w_2^* + 6(1 - w_1^* - w_2^*) = 6,$$

to find the optimal values of w_1^*, w_2^*, and $w_3^* = 1 - w_1^* - w_2^*$.

Finally, calculate the minimized value of the portfolio's standard deviation,

$$\sigma_P^* = \sqrt{64(w_1^*)^2 + 16(w_2^*)^2 + 36(w_3^*)^2}.$$

How does this minimized standard deviation compare to standard deviation $\sigma_3 = 6$ that an investor would have to accept by holding asset 3 alone?