Problem Set 10

$Y_0 =$ initial wealth

$a =$ amount allocated to stocks

$\tilde{r} =$ random return on stocks

$r_f =$ risk-free return

$\tilde{Y}_1 =$ terminal wealth

$$\begin{aligned}\tilde{Y}_1 &= (1 + r_f)(Y_0 - a) + a(1 + \tilde{r}) \\ &= Y_0(1 + r_f) + a(\tilde{r} - r_f)\end{aligned}$$

Problem Set 10

The investor chooses a to maximize expected utility:

$$\max_a E[u(\tilde{Y}_1)] = \max_a E\{u[Y_0(1+r_f) + a(\tilde{r} - r_f)]\}$$

The first-order condition is

$$E\{u'[Y_0(1+r_f) + a^*(\tilde{r} - r_f)](\tilde{r} - r_f)\} = 0.$$

Note: we are allowing the investor to sell stocks short ($a^* < 0$) or to buy stocks on margin ($a^* > Y_0$) if he or she desires.

Problem Set 10, Question 1

As an example, suppose $u(Y) = \ln(Y)$, as suggested by Daniel Bernoulli. Recall that for this utility function, $u'(Y) = 1/Y$. Then assume that stock returns can either be good or bad:

$$\tilde{r} = \begin{cases} r_G & \text{with probability } \pi \\ r_B & \text{with probability } 1-\pi \end{cases}$$

where $r_G > r_f > r_B$ defines the "good" and "bad" states and

$$\pi r_G + (1-\pi) r_B > r_f,$$

so that $E(\tilde{r}) > r_f$ and the investor will choose $a^* > 0$.

Problem Set 10, Question 1

The problem

$$\max_a E\{u[Y_0(1+r_f)+a(\tilde{r}-r_f)]\}$$

specializes to

$$\begin{aligned}\max_a \quad & \pi \ln[Y_0(1+r_f)+a(r_G-r_f)] \\ & +(1-\pi)\ln[Y_0(1+r_f)+a(r_B-r_f)]\end{aligned}$$

Problem Set 10, Question 1

$$\max_a \quad \pi \ln[Y_0(1+r_f) + a(r_G - r_f)] \\ + (1-\pi)\ln[Y_0(1+r_f) + a(r_B - r_f)]$$

Now set $Y_0 = 100$, $r_f = 0.10$, $r_G = 0.30$, $r_B = 0.05$, and $\pi = 1 - \pi = 1/2$. The problem becomes

$$\max_a \, (1/2)\ln(110 + 0.20a) + (1/2)\ln(110 - 0.05a)$$

To find a^*, differentiate with respect to a using the chain rule and set the result equal to zero.

Problem Set 10, Question 1

$$\max_a \; (1/2)\ln(110 + 0.20a) + (1/2)\ln(110 - 0.05a)$$

To find a^*, differentiate with respect to a using the chain rule and set the result equal to zero.

$$\frac{(1/2)(0.20)}{110 + 0.20a^*} - \frac{(1/2)(0.05)}{110 - 0.05a^*} = 0$$

Problem Set 10, Question 1

$$\frac{(1/2)(0.20)}{110 + 0.20a^*} - \frac{(1/2)(0.05)}{110 - 0.05a^*} = 0$$

$$\frac{0.20}{110 + 0.20a^*} = \frac{0.05}{110 - 0.05a^*}$$

$$0.20(110 - 0.05a^*) = 0.05(110 + 0.20a^*)$$

$$22 - 0.01a^* = 5.5 + 0.01a^*$$

a^* will turn out to be (much) larger than $Y_0 = 100$

Problem Set 10, Question 1

a^* will turn out to be (much) larger than $Y_0 = 100$

That is, the investor will find it optimal to buy stocks on margin.

This partly because an investor with log utility is not very risk averse.

But it is mainly because, with $r_G = 0.30$ and $r_B = 0.05$, stocks are not very risky in this example.

Problem Set 10, Question 1

You can check your answer by substituting $Y_0 = 100$, $r_f = 0.10$, $r_G = 0.30$, $r_B = 0.05$, and $\pi = 1 - \pi = 1/2$ into the general solution we derived in class:

$$\frac{a^*}{Y_0} = -\frac{(1+r_f)[\pi(r_G - r_f) + (1-\pi)(r_B - r_f)]}{(r_G - r_f)(r_B - r_f)},$$

Problem Set 10, Question 2

Let's generalize our previous example with logarithmic utility to the case where

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma},$$

with $\gamma > 0$. For this Bernoulli utility function, the coefficient of relative risk aversion is constant and equal to γ. The specific setting $\gamma = 1$ takes us back to the case with logarithmic utility.

Problem Set 10, Question 2

With $\gamma = 2$ instead:

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1-\gamma} = \frac{Y^{-1} - 1}{-1} = 1 - Y^{-1}$$

Problem Set 10, Question 2

And with $Y_0 = 100$, $r_G = 0.30$, $r_B = 0.05$, $r_f = 0.10$, $\pi = 1 - \pi = 1/2$, and $\gamma = 2$ so that $u(Y) = 1 - Y^{-1}$

$$\max_a \; (1/2)\{1 - [(1 + r_f)Y_0 + a(r_G - r_f)]^{-1}\} \\ + (1/2)\{1 - [(1 + r_f)Y_0 + a(r_B - r_f)]^{-1}\}$$

$$\max_a \; (1/2)[1 - (110 + 0.20a)^{-1}] + (1/2)[1 - (110 - 0.05a)^{-1}]$$

Problem Set 10, Question 2

$$\max_{a} \ (1/2)[1-(110+0.20a)^{-1}]+(1/2)[1-(110-0.05a)^{-1}]$$

$$\frac{(1/2)(0.20)}{(110+0.20a^*)^2}+\frac{(1/2)(-0.05)}{(110-0.05a^*)^2}=0$$

Problem Set 10, Question 2

$$\frac{(1/2)(0.20)}{(110+0.20a^*)^2}+\frac{(1/2)(-0.05)}{(110-0.05a^*)^2}=0.$$

$$\frac{(1/2)(0.20)}{(110+0.20a^*)^2}=\frac{(1/2)(0.05)}{(110-0.05a^*)^2}$$

$$4=\frac{(110+0.20a^*)^2}{(110-0.05a^*)^2}=\left(\frac{110+0.20a^*}{110-0.05a^*}\right)^2$$

Problem Set 10, Question 2

$$4 = \left(\frac{110 + 0.20a^*}{110 - 0.05a^*} \right)^2$$

$$2 = \frac{110 + 0.20a^*}{110 - 0.05a^*}$$

Strictly speaking, here, we are assuming that $110 + 0.20a^*$ and $110 - 0.05a^*$ are both positive. In practice, these restrictions will be "enforced" because stockbrokers won't allow the investor to take a short position that leads to bankruptcy in the good state or a leveraged long position that leads to bankruptcy in the bad state.

Problem Set 10, Question 2

$$2 = \frac{110 + 0.20a^*}{110 - 0.05a^*}$$

From here, you can solve for a^* and confirm that the amount allocated to stocks by the investor with $\gamma = 2$ is smaller than the amount allocated to stocks by the investor with $\gamma = 1$.