Solutions to Problem Set 9

ECON 337901 - Financial Economics — Peter Ireland
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For Extra Practice - Not Collected or Graded

1. Ordinal Utility

The Lagrangian

$$L = c_a^{\alpha} c_b^{1-\alpha} + \lambda(Y - p_a c_a - p_b c_b)$$

leads to the first-order conditions

$$\alpha (c_a^*)^{\alpha-1} (c_b^*)^{1-\alpha} - \lambda^* p_a = 0$$

and

$$(1-\alpha)(c_a^*)^{\alpha}(c_b^*)^{-\alpha} - \lambda^* p_b = 0.$$

Together with the budget constraint

$$Y = p_a c_a^* + p_b c_b^*$$

the first-order conditions form a system of three equations in the three unknowns: c_a^*, c_b^*, and λ^*.

There are a variety of ways to solve this three-equation system, but one is to divide the first first-order condition by the second to obtain

$$\frac{\alpha c_b^*}{(1-\alpha)c_a^*} = \frac{p_a}{p_b}$$

or

$$\alpha c_b^* = (1-\alpha)\left(\frac{p_a}{p_b}\right) c_a^*.$$

Then, rewrite the budget constraint as

$$c_a^* = \frac{Y}{p_a} - \left(\frac{p_b}{p_a}\right) c_b^*,$$

and substitute this expression into the one just before it to get

$$\alpha c_b^* = (1-\alpha)\left(\frac{p_a}{p_b}\right)\left[\frac{Y}{p_a} - \left(\frac{p_b}{p_a}\right) c_b^*\right] = (1-\alpha)\left(\frac{Y}{p_b}\right) - (1-\alpha)c_b^*$$

or, more simply,

$$c_b^* = \frac{(1-\alpha)Y}{p_b}.$$

Finally, use this solution for c_b^* together with budget constraint again to obtain

$$c_a^* = \frac{Y}{p_a} - \left(\frac{p_b}{p_a}\right) c_b^* = \frac{Y}{p_a} - \left(\frac{p_b}{p_a}\right)\left[\frac{(1-\alpha)Y}{p_b}\right] = \frac{Y}{p_a} - (1-\alpha)\left(\frac{Y}{p_a}\right)$$

or, more simply,

$$c_a^* = \frac{\alpha Y}{p_a}.$$

As in question 1 from problem set 2, the consumer finds it optimal to spend the fraction α of his or her income on apples and the fraction $1-\alpha$ on bananas. This is not a coincidence. Taking the natural log of the utility function used here yields

$$\ln(c_a^{\alpha} c_b^{1-\alpha}) = \alpha \ln(c_a) + (1-\alpha)\ln(c_b)$$

which coincides with the utility function used in the previous problem. Since the natural logarithm is a strictly increasing function, the two utility functions represent exactly the same underlying preference ordering.

2. Expected Utility and Aversion to Risk

With von Neumann-Morgenstern expected utility function

$$U(x, y, \pi) = \pi u(W_0 + x) + (1-\pi)u(W_0 + y) = \pi\left[\frac{(W_0+x)^{1-\gamma}}{1-\gamma}\right] + (1-\pi)\left[\frac{(W_0+y)^{1-\gamma}}{1-\gamma}\right]$$

and $W_0 = 10$, the table below compares the three lotteries $(x, y, \pi) = (5, 0, 1/2)$, $(x, y, \pi) = (2.5, 0, 1)$ and $(x, y, \pi) = (2, 0, 1)$ when $\gamma = 1/2$, $\gamma = 2$ and $\gamma = 3$.

γ	$U(5,0,1/2)$	$U(2.5,0,1)$	$U(2,0,1)$
1/2	7.0353	7.0711	6.9282
2	−0.0833	−0.0800	−0.0833
3	−0.0036	−0.0032	−0.0035

For all values of γ, the investor always prefers getting the average of 2.5 for sure to the alternative of 5 with probability 1/2 and 0 with probability 1/2. This first set of comparisons shows us once again how the concavity of the Bernoulli utility function represents the investor's aversion to risk. On the other hand, even a risk averse investor will be willing to accept gambles when the safer alternative offers less than the expected value of the bet. In this case, the investor with $\gamma = 1/2$ prefers the risky bet to receiving 2 for sure; the investor with $\gamma = 2$ is indifferent between the two options, and the investor with $\gamma = 3$ prefers receiving 2 for sure. This second set of comparisons suggests that γ is a measure of risk aversion, with higher values of γ implying more risk averse behavior.