Solutions to Problem Set 7

ECON 337901 - Financial Economics — Peter Ireland
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For Extra Practice - Not Collected or Graded

1. Stocks and Bonds as Portfolios of Contingent Claims

There are two periods, $t = 0$ and $t = 1$, and two possible states at $t = 1$: a "good" state that occurs with probability $\pi = 1/2$ and a "bad" state that occurs with probability $1 - \pi = 1/2$. At first, investors trade two contingent claims. A contingent claim for the good state sells for $q^G = 0.30$ at $t = 0$ and pays off one dollar in the good state at $t = 1$ and zero in the bad state at $t = 1$. A contingent claim for the bad state sells for $q^B = 0.60$ at $t = 0$ and pays off one dollar in the bad state at $t = 1$ and zero in the good state at $t = 1$.

Now suppose two new assets are introduced. The first, a stock, pays a dividend of $d^G = 3$ in the good state at $t = 1$ and $d^B = 1$ in the bad state at $t = 1$. The second, a bond, pays off one dollar in both states, the good and bad, at $t = 1$.

a. To replicate the payoffs from the stock, the investor should buy $s^G = 3$ contingent claims for the good state and $s^B = 1$ contingent claim for the bad state. No arbitrage requires that the price of the stock at $t = 0$ equal the cost of assembling this portfolio of contingent claims. Therefore, given the contingent claims prices $q^G = 0.30$ and $q^B = 0.60$, the stock price must be

$$q^s = s^G q^G + s^B q^B = 3(0.30) + 1(0.60) = 0.90 + 0.60 = 1.50.$$

b. To replicate the payoffs from the bond, the investor should buy $s^G = 1$ contingent claim for the good state and $s^B = 1$ contingent claim for the bad state. No arbitrage requires that the price of the bond at $t = 0$ equal the cost of assembling this portfolio of contingent claims. Therefore, given the contingent claims prices $q^G = 0.30$ and $q^B = 0.60$, the bond price must be

$$q^b = s^G q^G + s^B q^B = 1(0.30) + 1(0.60) = 0.30 + 0.60 = 0.90.$$

c. Since the bond that pays off a dollar for sure at $t = 1$ sells for $q^b = 0.90$ at $t = 0$, the interest rate is

$$r^b = \frac{1 - q^b}{q^b} = \frac{1 - 0.90}{0.90} = \frac{0.10}{0.90} = 0.111 = 11.1 \text{ percent}.$$

2. Contingent Claims as Portfolios of Stocks and Bonds

Now, we're assuming that instead of contingent claims, only the stock and bond described in question one trade in this economy. Exactly as above, the stock sells for $q^s = 1.50$ at $t = 0$ and pays a dividend $d^G = 3$ in the good state at $t = 1$ and $d^B = 1$ in the bad state at $t = 1$. The bond sells for $q^b = 0.90$ at $t = 0$ and pays off one dollar for sure in both states, the good and bad, at $t = 1$.

a. To create "synthetic" contingent claim for the good state, an investor will form a portfolio consisting of s shares of stock and b bonds. To match the claim's payoff of one dollar in the good state, s and b must satisfy

$$1 = d^G s + b = 3s + b.$$

And to match the claim's payoff of zero in the bad state, s and b must satisfy

$$0 = d^B s + b = s + b.$$

Subtracting the second equation from the first yields

$$1 = 2s,$$

which shows that $s = 1/2$. Substituting this solution of $s = 1/2$ into either equation then shows that

$$b = -1/2.$$

Evidently, assembling this portfolio involves taking a long position in the stock and a short position in the bond. No arbitrage then requires that the price of the contingent claim for the good state at $t = 0$ equal to cost of assembling this portfolio of the stock and the bond. Given the stock and bond prices $q^s = 1.50$ and $q^b = 0.90$, the contingent claim price must be

$$q^G = sq^s + bq^b = (1/2)(1.50) - (1/2)(0.90) = 0.75 - 0.45 = 0.30.$$

Of course, this value of q^G implied by the stock and bond prices is the same value that we originally assumed in question 1.

b. Similarly, to create synthetic contingent claim for the bad state, an investor will form a portfolio consisting of s shares of stock and b bonds. To match the claim's payoff of zero in the good state, s and b must satisfy

$$0 = d^G s + b = 3s + b.$$

And to match the claim's payoff of one dollar in the bad state, s and b must satisfy

$$1 = d^B s + b = s + b.$$

Subtracting the second equation from the first yields

$$-1 = 2s,$$

which shows that $s = -1/2$. Substituting this solution of $s = -1/2$ into either equation then shows that

$$b = 3/2.$$

Evidently, assembling this portfolio involves taking a short position in the stock and a long position in the stock. No arbitrage then requires that the price of the contingent claim for the good state at $t = 0$ equal to cost of assembling this portfolio of the stock and the bond. Given the stock and bond prices $q^s = 1.50$ and $q^b = 0.90$, the contingent claim price must be

$$q^B = sq^s + bq^b = -(1/2)(1.50) + (3/2)(0.90) = -0.75 + 1.35 = 0.60.$$

Once again, of course, this value of q^B implied by the stock and bond prices coincides with the value that we originally assumed in question 1.

3. Arrow-Debreu No Arbitrage Pricing

Finally, two more new assets are introduced. The first is another stock, which pays a dividend of $d_2^G = 4$ in the good state at $t = 1$ and a dividend of $d_2^B = 2$ in the bad state at $t = 1$. The second is an "exotic" asset, which makes a payment of 100 in the bad state at $t = 1$ but requires the holder to *make* a payment of 100 in the good state at $t = 1$.

a. To replicate the payoffs from the second stock, an investor can buy four contingent claims for the good state and two contingent claims for the bad state. No arbitrage then requires that the price of the second stock equal the cost of assembling the portfolio of contingent claims. Therefore,

$$q_2^s = 4(0.30) + 2(0.60) = 1.20 + 1.20 = 2.40.$$

Notice, too, that the payoffs from the second stock could also be replicated by a portfolio that consists of one share of first stock and one bond. The cost of assembling this portfolio equals the price of the first stock $q^s = 1.50$ plus the price of the bond $q^b = 0.90$; no arbitrage then leads to the same conclusion that the price of the second stock should be $q_2^s = 2.40$.

b. To replicate the cash flows from the exotic asset, the investor can buy 100 contingent claims for the bad state and sell short 100 claims for the good state. No arbitrage then requires that the price of the exotic asset equal to cost of assembling the portfolio of contingent claims. Therefore

$$q^A = -100(0.30) + 100(0.60) = -30 + 60 = 30.$$

If you really felt like it, you could also deduce that the cash flows from this exotic asset could be replicated by a portfolio assembled by selling short 100 shares of the first stock and buying 200 bonds. No arbitrage then leads to the same conclusion that the price of the exotic asset should be $q^A = -100(1.50) + 200(0.90) = 30$.